



Dendritic Solidification of Binary Mixtures of Metals under the action of Magnetic Field: Modeling, Mathematical and Numerical Analysis

Amer Rasheed

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Rasheed Amer

Solidification Dendritique de Mélanges Binaires de Métaux sous l'Action de Champs Magnétiques: Modélisation, Analyse Mathématique et Numérique

Soutenue le 14.10.2010 devant la commission d'examen

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Abstract

In order to understand the behavior of materials in the presence of impurities during the solidification process, it was required to develop appropriate methodologies for an analysis and an effective control of the topological changes of the microstructures (e.g., the formation of dendrites) during the different phases of transformation. The objective of this thesis is to build a relevant model of solidification of binary alloys under the action of magnetic fields, to analyze the obtained systems, from a theoretical and a numerical point of view, and finally, to develop an optimal control method to control the dynamics of the solidification front by the action of magnetic fields.

Initially, we have described the physics of the problem and the fundamental laws necessary for modeling, then we built a new model of phase field, which takes into account the influence of the action of magnetic field on the movement of the solidification front. The model thus developed is characterized by the coupling of three systems: one of magnetohydrodynamic type, a second of Boettinger Warren-convection type (representing the evolution of the solidification front and the concentration of the binary mixture) and a third representing the evolution of the temperature. The equations of the complete system describing the model, in a domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, are time-dependent, nonlinear, coupled and anisotropic. In a second part, we have performed the theoretical analysis of the model in the two-dimensional, isothermal and isotropic case. We have obtained results of existence, regularity, stability and uniqueness of the solution, under certain conditions on nonlinear operators of the system. Finally, we have developed a nonlinear optimal control method : the magnetic field (which acts multiplicatively) plays the role of the control, and the observation is the desired state of the dynamics of the front. We have proved the existence of an optimal solution and obtained the sensitivity of the operator solution and the optimality conditions by introducing an adjoint problem.

The theoretical part of the thesis is supplemented by an important numerical work. The analysis and numerical simulations have been conducted on the complete two-dimensional nonlinear (isotropic and anisotropic) problem. We used, for

discretization, the method of lines which consists to consider separately the spatial and temporal discretization. The spatial discretization is performed by using a mixed finite elements scheme and the resolution of the obtained algebraic differential system is performed by using the DASSL solver. The discretization of the domain is performed by unstructured triangular meshes. In the realistic case, they correspond to a non-uniform mesh that is very fine in area of the dendrite and at the interface. We have obtained error estimates for the different state variables of the model and analyzed the robustness and stability of the approximation schemes. This numerical code has been validated on various examples, and gives excellent results. Then we have used the code to treat a realistic problem, namely the dendritic solidification of a binary alloy Nickel-Copper, and to analyze the influence of magnetic fields on the development of dendrites. The results show the effectiveness of the approach to reproduce the experimental observations.

Résumé

La compréhension du comportement des matériaux en présence d'impuretés, durant le processus de solidification, nécessite le développement de méthodologies appropriées pour une analyse et un contrôle efficace des changements topologiques des microstructures (par exemple, la formation des dendrites) au cours des différentes phases de transformation. L'objectif de cette thèse est de construire un modèle pertinent de solidification d'alliages binaires sous l'action de champs magnétiques, d'analyser les systèmes issus du modèle mathématique ainsi développé, d'un point de vue théorique et numérique, et enfin de développer une méthode de contrôle optimal afin de contrôler la dynamique du front de solidification par l'action du champs magnétiques.

Dans un premier temps, nous avons décrit la physique du problème et les lois fondamentales nécessaires à la modélisation, puis nous avons construit un nouveau modèle de champ de phase, qui tient compte de l'influence de l'action du champ magnétique sur le mouvement du front de solidification. Le modèle ainsi développé est caractérisé par le couplage de trois systèmes : un de type magnétohydrodynamique, un second de type Warren-Boettinger avec convection (représentant l'évolution du front de solidification et la concentration du mélange binaire) et un troisième représentant l'évolution de la température. Les équations du système complet décrivant le modèle, dans un domaine $\Omega \subset \mathbb{R}^n$, $n \leq 3$, sont évolutives, non linéaires, couplées et anisotropes. Dans une seconde partie, nous avons effectué l'analyse théorique du modèle développé dans le cas isotherme et isotrope en dimension deux. Nous avons obtenu des résultats d'existence, de régularité, de stabilité et d'unicité d'une solution, sous certaines conditions sur des opérateurs non linéaires du système. Enfin, nous avons développé une méthode de contrôle optimal non linéaire : le champ magnétique (qui intervient sous forme multiplicative) joue le rôle de contrôle, et l'observation est l'état désiré de la dynamique du front. Nous avons démontré l'existence d'une solution optimale et obtenu la sensibilité de l'opérateur solution et les conditions d'optimalité en introduisant un problème adjoint.

Cette partie théorique de la thèse est complétée par un important travail numérique. L'analyse et les simulations numériques ont été menées sur le problème complet bi-dimensionnel non linéaire (isotrope et anisotrope). Nous avons utilisé pour la discrétisation la méthode des lignes qui consiste à considérer séparément la discrétisation temporelle et spatiale. La discrétisation spatiale est effectuée par un schéma d'éléments finis mixtes et le système différentiel algébrique obtenu est résolu par l'utilisation du solveur DASSL. La discrétisation du domaine est effectuée par des mailles triangulaires non structurées. Dans le cas réaliste, elles correspondent à un maillage non uniforme et très fin dans la zone de la dentrite et au niveau de l'interface. Nous avons obtenu des estimations d'erreur pour les différentes variables d'état du modèle et analysé la robustesse et la stabilité des schémas d'approximation. Ce code numérique a été validé sur différents exemples, et donne d'excellents résultats. Ensuite, nous avons exploité le code pour traiter un problème réaliste, à savoir la solidification dendritique d'un alliage binaire Nickel-Cuivre, et analyser l'influence de champs magnétiques sur l'évolution des dendrites. Les résultats obtenus montrent l'efficacité de l'approche à reproduire les observations expérimentales.

Introduction

La solidification (ou congélation) est le processus par lequel un métal pur ou un mélange de deux ou plusieurs métaux sous forme liquide se transforme en solide par refroidissement (cas classique), par augmentation de la pression, ou bien par une combinaison des deux. En présence d'impuretés dans les métaux, lors du processus de solidification, des cristaux ramifiés en forme d'arbre, appelés dendrites, commencent à se générer autour de ces impuretés. Les microstructures des dendrites ainsi générées durant ce processus, déterminent les futures propriétés du matériau (solidifié). De plus, ce front de solidification est en général instable (très sensible aux variations de gradient de température ou de composition chimique, ceux-ci jouant un rôle important dans la croissance). Par conséquent, l'observation et l'analyse de ce front de solidification ont un grand intérêt scientifique et industriel. Afin d'améliorer la qualité et les propriétés des mélanges, le défi industriel majeur réside dans la possibilité de contrôler la structure du métal ainsi que ses défauts. C'est pour cette raison que les scientifiques tentent de maîtriser la croissance et la structure des dendrites pendant le processus de solidification, afin d'obtenir les propriétés désirées pour les métaux (ou alliages) considérés.

Dans la littérature, il existe deux types d'approche pour modéliser ce phénomène de solidification. La première est le modèle de surface libre de Stefan entre les phases liquide et solide, qui tient compte de la diffusion de chaleur dans chaque phase et l'échange de chaleur latente à l'interface des phases. Les équations satisfaites par les variables thermodynamiques, comme la température et la composition du système, sont formulées et résolues de façon indépendante pour chaque phase. Les conditions aux limites au niveau de l'interface solide-liquide sont basées classiquement sur les lois de conservation (de l'énergie). Pour plus de détails, on peut consulter, par exemple, le livre de A. Visintin [9].

La seconde approche est la méthode de champ de phase, qui, contrairement à l'approche classique de Stefan (qui impose beaucoup de contraintes), traite le système dans son ensemble. Dans cette approche, les équations décrivent à la fois les

phases liquide et solide et l'interface liquide/solide, par l'introduction d'une variable d'état supplémentaire dite de champ de phase ψ , un paramètre d'ordre abstrait qui représente la transition de phase à chaque point de l'espace et à chaque instant ⁽¹⁾. Cette nouvelle variable ψ prend des valeurs constantes dans la phase liquide, par exemple 0, et dans la phase solide, par exemple 1, et permet une transition entre ces deux phases avec une variation régulière mais rapide entre 0 et 1. Cette modélisation évite en particulier un traitement numérique spécifique de l'interface solide/liquide, tout en reproduisant les principaux mécanismes physiques de la transition de phase entre les deux phases. En effet, l'interface apparaît naturellement dans ce modèle par l'introduction de l'équation de champ de phase, ce qui permet ainsi la résolution numérique du modèle à l'aide de schémas classiques. En outre, cette approche permet également de prendre en compte naturellement différents phénomènes physiques, tels que l'élasticité ou l'électromagnétisme.

Les modèles de champ de phase sont devenus un outil important pour simuler, lors du processus de solidification d'alliage binaire, la formation et la croissance des dendrites. Ils ont fait l'objet de très nombreux travaux aussi bien d'un point de vue mathématique que numérique, voir par exemple, A. A. Wheeler et al. [1], A. Belmiloudi [6]-[8], G. Caginalp [19], J. Rappaz [34], P. Laurencot [49] S. L. Wang [57]. On peut noter l'existence de solutions analytiques pour ce type de modèle, mais cela reste limité à des cas très simples. Dans le cas de situations réalistes où le système est fortement non linéaire et très complexe, la simulation numérique est un outil nécessaire, voire indispensable; il joue un rôle important dans la compréhension et l'analyse de la formation des microstructures des dendrites. Dans ce cadre, on peut citer différents travaux de simulations de la croissance dendritique de métaux purs ou mélangés, par exemple, les travaux de B. Kaouil et al. [10], D. Kessler [14], J. A. Warren and W. J. Boettinger [24], J. C Ramirez and C. Beckermann et al. [25]-[26], M. Grujicic et al. [39], O. Kruger [46], R. Kobayashi [53], T. Takaki et al. [58]. De plus, ces dix dernières années, la méthode de champ de phase a été étendue pour inclure l'effet de la convection sur la croissance des dendrites. Cela a été motivé par le fait que pendant des expériences de solidification, on a observé un impact significatif du mouvement dans le liquide sur la formation et l'évolution de la microstructure dendritique. Pour des travaux sur les simulations des modèles de champs de phase intégrant la convection, on peut citer par exemple dans le cas de modèles pour la solidification d'un métal pur: D. M. Anderson et al. [15] qui ont développé un modèle en

⁽¹⁾ Ce paramètre peut être comparé à la fonction indicatrice de phase utilisée dans les techniques numériques traitant les interfaces comme des surfaces de discontinuité.

utilisant des équations de Navier-Stokes et en supposant que la viscosité et la densité sont des fonctions dépendant du champ de phase (les deux phases sont traitées comme deux fluides); R. Tonhardt et G. Amberg [56], et Tong X. et al. [61] qui ont donné des modèles par l'introduction de la convection naturelle en utilisant les équations de type Navier-Stokes et en forçant la vitesse à être nulle dans la phase solide. Pour d'autres modèles, on peut citer, par exemple, les travaux de N. Al-Rawahi et G. Tryggvason [44, 45], et E. Bansch et A. Schmidt [16].

Récemment, il a été observé expérimentalement que le mouvement du fluide et sa direction peuvent être contrôlés en appliquant des champs magnétiques ou des courants électriques pendant le processus de solidification, afin d'améliorer la qualité et les propriétés des métaux. Par exemple, Mingjun Li et al. [40]-[41] ont montré expérimentalement que les dendrites secondaires dans le matériau peuvent devenir plus fines, plus homogènes et équiaxes par l'application de champs magnétiques ou de courants électriques durant la solidification. Malheureusement, ils n'ont pas discuté l'effet du champ magnétique ou du courant électrique sur la structure elle-même des dendrites. Il est donc nécessaire maintenant d'analyser les effets et les influences du champ magnétique (ou du courant électrique) sur la dynamique et la structure des dendrites. D'autres applications de l'influence des champs magnétiques (ou courants électriques) sur le comportement des matériaux, ont été étudiées : on peut citer, par exemple, pour des écoulements MHD, H. Ben Hadid et al. [21]-[22]; dans le cadre des semi-conducteurs et la croissance des cristaux, A. Belmiloudi [8], M. Gunzberger et al. [38], M. Watanabe et al. [42], V. Galindo et al. [59] et pour les processus de solidification, J. K. Roplekar et J. A. Dantzig [29], J. Rappaz et R. Touzani [35], P. J. Prescott [48].

Dans cette thèse, nous avons développé un modèle de champ de phase pour analyser l'effet du champ magnétique sur la dynamique des dendrites lors de la solidification d'un mélange binaire. Considérons un alliage de deux composants A et B dans un domaine spatial Ω , le système est caractérisé par un couplage entre l'équation de la concentration relative du composant B avec le respect du mélange, l'équation du champ de phase, l'équation de l'énergie et le système d'écoulement magnétohydrodynamique qui décrit le mouvement dans la phase liquide sous l'effet du champ magnétique dans un environnement non-isotherme. Le point de départ de notre travail est le modèle à deux dimensions de J. A. Warren et W. J. Boettinger [24], dont les variables d'état sont la fonction de champ de phase et la concentration. Dans le présent travail, nous avons tout d'abord généralisé les modèles de J. A. Warren et W.

J. Boettinger [24] et de M. Grujicic et al. [39] (en tenant compte des différentes observations expérimentales, citées précédemment), en incluant l'effet de la convection dans les équations de champ de phase, de la concentration et de l'énergie ainsi qu'en introduisant un système d'écoulement magnétohydrodynamique qui tient compte des effets du champ magnétique. Ensuite, nous avons discuté différentes situations et étudié théoriquement et numériquement le modèle développé précédemment. Plus précisément, le travail de la thèse est organisé comme suit.

Dans le chapitre 1, nous décrivons tout d'abord la physique du problème, ensuite nous donnons les lois fondamentales nécessaires à la modélisation et enfin nous construisons un nouveau modèle de champ de phase, dans un environnement non-isotherme, qui tient compte de l'influence de l'action du champ magnétique sur le mouvement du front de solidification. Le modèle ainsi développé est caractérisé par le couplage de trois systèmes. Le premier est un système de type magnétohydrodynamique défini par des équations de type Navier-Stokes incompressible et des approximations de Boussinesq couplées à l'électromagnétisme (qui représente le mouvement dans la phase liquide sous l'action d'un champ magnétique). Le second est un système de type Warren-Boettinger avec convection (non linéaire de type transport-diffusion) qui représente l'évolution de la fonction de champ de phase et de la concentration au cours du processus de solidification. Le troisième est un système d'énergie (non linéaire de type transport-diffusion) qui représente l'évolution de température durant le processus de solidification. Les équations du système complet décrivant le modèle, dans un domaine $\Omega \subset \mathbb{R}^n$, $n \leq 3$, sont évolutives, non linéaires, couplées et anisotrope. Les variables d'état du modèle sont \mathbf{u}, p, ψ, c et T , où, pour x dans Ω et à l'instant t , $\mathbf{u}(\mathbf{x}, t)$ et $p(\mathbf{x}, t)$ représentent la vitesse et la pression dans le système de type magnétohydrodynamique; $\psi(\mathbf{x}, t)$ est la variable de champ de phase dont la valeur varie entre 0 (quand le système est dans une phase solide) et 1 (quand le système est dans une phase liquide), sur une fine couche qui sépare les deux phases; $c(\mathbf{x}, t)$ représente la concentration relative qui varie également entre 0 et 1; $T(\mathbf{x}, t)$ représente la température du système. Nous avons examiné différents cas particuliers de notre modèle à savoir, le cas isotrope, l'environnement isotherme, le cas anisotrope-isotherme et le cas de la dimension deux.

Dans le chapitre 2, nous avons effectué l'analyse mathématique du modèle développé dans le cas isotherme et isotrope en dimension deux. Nous avons obtenu des résultats d'existence, de régularité, de stabilité et d'unicité d'une solution (\mathbf{u}, p, ψ, c) , sous certaines conditions de Lipschitz et de bornitude sur des opérateurs non linéaires du système. Le résultat d'existence est prouvé par l'utilisation d'une méthode de type

Faedo-Galerkin. Pour obtenir l'unicité et la stabilité en fonction des données, nous avons été amené à obtenir une régularité très fine de la solution. Plus précisément, nous avons prouvé, entre autres, la régularité suivante

$$\begin{aligned} \mathbf{u} &\in L^2\left(0, T_f, (H^2(\Omega))^2\right) \cap L^\infty\left(0, T_f, (H_0^1(\Omega))^2\right), \quad \frac{\partial \mathbf{u}}{\partial t} \in L^2\left(0, T_f; (L^2(\Omega))^2\right), \\ \psi &\in L^2\left(0, T_f, H^3(\Omega)\right) \cap L^\infty\left(0, T_f, H^2(\Omega)\right), \quad \frac{\partial \psi}{\partial t} \in L^2(0, T_f; H^1(\Omega)), \\ c &\in L^2\left(0, T_f, H^2(\Omega)\right) \cap L^\infty\left(0, T_f, H^1(\Omega)\right), \quad \frac{\partial c}{\partial t} \in L^2(0, T_f; L^2(\Omega)), \end{aligned}$$

où T_f est le temps final et $\Omega \subset \mathbb{R}^2$.

Dans le chapitre 3, nous avons effectué l'analyse numérique des modèles développés dans le cas isotherme et bi-dimensionnel c'est-à-dire le cas isotherme-isotrope (TDII) et le cas isotherme-anisotrope (TDIA). Nous avons développé un schéma numérique pour résoudre ces modèles et étudié la convergence et la stabilité du schéma d'approximation pour ces deux modèles. La discrétisation en espace est effectuée en utilisant des éléments finis mixtes et nous avons utilisé le solveur DASSL pour intégrer numériquement le système différentiel non linéaire obtenu après discrétisation. Pour étudier la convergence espace-temps du schéma numérique développé, nous avons considéré différents types d'éléments finis mixtes. Pour étudier la stabilité du schéma, nous avons généré des perturbations dans les équations du modèle par l'introduction d'une fonction aléatoire, dont les valeurs varient entre 0 et 1, multipliée par un paramètre qui contrôle le pourcentage d'erreur aléatoire. Cette stabilité a été étudiée en augmentant progressivement le pourcentage d'erreur jusqu'à 40%. La convergence, les estimations d'erreurs et la stabilité du schéma ont été validées numériquement sur différents exemples. Cette analyse a montré, entre autres, la stabilité du schéma et l'adéquation entre les estimations d'erreurs numériques et les estimations d'erreurs théoriques. Pour mettre en œuvre le schéma numérique, nous avons utilisé les logiciels COMSOL Multiphysics version 3.4 et MatLab version (2007a). Le couplage entre Matlab et Comsol s'est avéré indispensable pour nous permettre d'une part d'introduire l'opérateur différentiel d'anisotropie et les termes aléatoires dans les modèles, d'autres part pour analyser la convergence et étudier les estimations d'erreurs espace-temps, et enfin pour mettre en œuvre le problème de contrôle qui est en cours de test et de validation.

Le but du chapitre 4 est d'analyser numériquement l'influence du champ magnétique sur la dynamique et la structure des dendrites durant la solidification de l'alliage binaire Nickel-Cuivre (Ni-Cu) dans le cas de données réalistes et du modèle complet (TDIA). Pour cela, nous avons tout d'abord adimensionalisé le système, et ensuite

réduit le modèle, par l'élimination du système de la magnétohydrodynamique et des termes de convection, afin de valider notre approche en comparant les résultats du modèle réduit avec ceux de Warren et Boettinger, et Grujicic et al. Pour analyser l'influence du champ magnétique, nous avons simulé dans un premier temps le modèle (TDIA) sans champ magnétique et ensuite, nous avons considéré l'influence de trois fonctions de champ magnétique à savoir deux champs magnétiques constants classiques d'angle 90° et 45° , et un champ magnétique variable. Nous n'avons pas observé d'influence significative du champ magnétique d'angle 90° sur la forme des dendrites, contrairement au champ magnétique d'angle 45° ; par contre le champ magnétique variable déforme considérablement la structure de la dentrite. Nous avons aussi observé que la direction du mouvement du liquide a été complètement modifiée sous l'action des champs magnétiques exercés. On peut conclure que l'application du champ magnétique influence considérablement la dynamique et la structure des dendrites, ainsi que la direction du mouvement du liquide. Ces observations sont en accord avec les études expérimentales développées, par exemple, par Prescott [48]. Cette dernière analyse a motivé le travail du chapitre 5.

Le chapitre 5 concerne le contrôle de la dynamique du front de solidification par l'action du champ magnétique, en utilisant la théorie du contrôle optimal, la fonction de champ magnétique jouant le rôle de variable de contrôle. Pour les problèmes de contrôle associés aux modèles de champ de phase, on peut citer K. H. Hoffman et al. [36], où les auteurs ont étudié le contrôle optimal de la solidification des matériaux purs; A. Belmiloudi et al. [6]-[8], où les auteurs ont étudié les problèmes de contrôle robuste et de stabilisation du front lors d'une solidification non isotherme de matériaux purs et d'une solidification isotherme d'alliages binaires, en tenant compte de l'influence des bruits et des fluctuations. Pour des problèmes de contrôle en utilisant le champ magnétique comme variable de contrôle, on peut citer, par exemple dans le cadre de matériaux semiconducteur fondu et le processus de croissance d'un cristal par méthode Czochralski, M. Gunzburger et al. [38], où les auteurs ont étudié le contrôle du gradient de température dans le cristal durant ce processus et A. Belmiloudi [8], où l'auteur a analysé la stabilisation de la dynamique durant ce processus en tenant compte des fluctuations et des impuretés du matériau.

Le problème de contrôle non linéaire traité dans cette thèse diffère des problèmes examinés par les auteurs cités précédemment, par la nature du système d'équations du modèle considéré qui inclut, en plus du système concentration-champ de phase, un système de type magnétohydrodynamique augmenté de nouveaux opérateurs non linéaires qui intègrent entre autres le champ magnétique \mathbf{B} sous la forme non linéaire

suivante $(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} + a(\psi)\mathbf{B}$ où \mathbf{u} est la vitesse et ψ est le paramètre de champ de phase. Le problème de contrôle a été formulé dans le cadre du modèle (TDII), ensuite l'existence d'une solution optimale a été analysée et les conditions d'optimalité ont été obtenues.

Chapter 1

Modeling Solidification and Melting Problems

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1.1 Introduction

Application of the phase-field methods and other diffusive interface methods of solidification has been limited to the problems where the transport of heat and/or solute is by diffusion only. But, when the solid crystal (or seed) grows in a solidification process, the size, morphology, growth rate etc. of the crystal may be affected by any motion in the melt. This motion in a melt is therefore important and cannot be ignored. Such melt flows may occur due to, for example, temperature gradients, concentration gradients, buoyancy-driven flow, natural convection due to the release of latent heat from the growing crystal etc. In this chapter we shall derive the model problem which describes the solidification of a binary alloy in the presence of flow in the liquid phase. The commencing point of the present work is the two dimensional model of solidification of binary alloy of J. A. Warren, W. J. Boettinger [24] and M. Grujicic [39]. In the present work this model will be extended subsequently to include the effects of convection in the phase-field, concentration and energy equations and also the equations of melt flow in the presence of a magnetic field, applied to the entire domain, will be included.

In the first section 1.1 we shall give the general physical laws used to derive the evolution equations of the phase field, concentration, energy density and melt flow. In the section 1.2 we shall derive the equation of phase field explicitly which is based on the entropy functional analogous to Wang et al.[57]. In the section 1.3 we shall give the derivation of evolution equations of concentration and energy which are derived using the theory of irreversible processes. And in the section 1.4, we shall derive the equations of the melt flow using the incompressible Navier-Stokes equations with the Boussinesq approximations and the entire set of model equations will be given in the section 1.5. Finally, we shall give the two dimensional isothermal and isotropic model in the section 1.6.

1.2 Physical Laws

Let Ω be a closed bounded region in \mathbb{R}^n , where n is the number of space dimension, with a piecewise smooth boundary $\Gamma = \partial\Omega$. Initially the region Ω is occupied by a binary alloy of the solute B in the solvent A, which is considered as incompressible electrically conducting fluid.

At time t , the position of the system is described by the phase field variable $\psi(\mathbf{x}, t)$ which takes values in the interval $[0, 1]$ where the values $\psi = 0$ and $\psi = 1$ correspond

to the pure solid and pure liquid phases respectively (see the Fig. 1.1), concentration $c(\mathbf{x}, t)$ which is the mole fraction of solute B in the solvent A, energy density $e(\mathbf{x}, t)$ and the velocity field $\mathbf{u}(\mathbf{x}, t)$. The governing equations for the and concentration

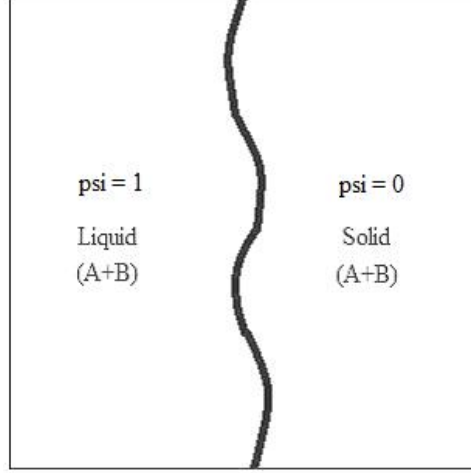


Figure 1.1: Solidification of a binary alloy.

$c(\mathbf{x}, t)$ will be derived by using conservation laws of the energy and concentration respectively as

$$\frac{De(\mathbf{x}, t)}{Dt} + \text{div}(\mathbf{J}_e) = 0, \quad (1.1)$$

$$\frac{Dc(\mathbf{x}, t)}{Dt} + \text{div}(\mathbf{J}_c) = 0, \quad (1.2)$$

where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material time derivative and \mathbf{J}_e and \mathbf{J}_c are the conserved fluxes of energy and concentration respectively. These equations depend on the entropy functional, denoted by $S(\psi, c, e)$, which will be used to construct the expressions for the fluxes of energy density and concentration respectively.

As the phase field variable $\psi(\mathbf{x}, t)$ is not a conserved quantity therefore the most appropriate form of the evolution equation for the phase field is (as in [24])

$$\frac{D\psi(\mathbf{x}, t)}{Dt} = M_\psi \frac{\delta S(\psi, c, e)}{\delta \psi}, \quad (1.3)$$

where $M_\psi > 0$ is the interfacial mobility parameter and operator δ denotes variational derivative and $S(\psi, c, e)$ is the entropy functional which will be given later. The phase field variable $\psi(\mathbf{x}, t)$ varies smoothly in the interval $(0, 1)$ and its value in the solid phase is 0 and in the liquid phase is 1.

The governing equation for the velocity field $\mathbf{u}(\mathbf{x}, t)$ will be derived by using the conservation of momentum and mass as

$$\rho \frac{D\mathbf{u}(\mathbf{x}, t)}{Dt} = \text{div}(\vec{\sigma}) + \rho \mathbf{B}_f, \quad (1.4)$$

$$\text{div}(\mathbf{u}(\mathbf{x}, t)) = 0, \quad (1.5)$$

where ρ is the density of the fluid, $\vec{\sigma}$ is the stress tensor, $\mathbf{u}(\mathbf{x}, t)$ is the velocity of the fluid and \mathbf{B}_f is the body force.

In the next section, the detailed derivation of the evolution equation for the phase-field variable $\psi(\mathbf{x}, t)$ is given.

1.3 Evolution Equation for Phase-Field Variable

To derive the evolution equation for the phase-field variable $\psi(\mathbf{x}, t)$ that indicates the phase of the material at each point (\mathbf{x}, t) , we demand that the entropy of an irreversible system always increases locally for a system where the internal energy and concentration are conserved, the entropy is represented by the functional [24]

$$S(\psi, c, e) = \int_{\Omega} \left(s(\psi, c, e) - \frac{\epsilon_{\theta}^2}{2} |\nabla \psi|^2 \right) d\mathbf{x}, \quad (1.6)$$

where $s(\psi, c, e)$ is an entropy density, $e(\mathbf{x}, t)$ is the internal energy, $\psi(\mathbf{x}, t)$ is the phase-field variable and $c(\mathbf{x}, t)$ is the mole fraction of solute B in the solvent A. The second term in the integrand is a gradient entropy term analogous to the gradient energy term in the free energy, where the parameter ϵ_{θ} is the interfacial energy parameter which represents the gradient corrections to the entropy density. Here, we have omitted the gradient corrections in the concentration $c(\mathbf{x}, t)$ and energy density $e(\mathbf{x}, t)$.

Now we need to take variational derivative of the functional $S(\psi, c, e)$ in the sense of distribution. Let X be a topological space and U is an open set in X . Then variational derivative of equation (1.6) at $\psi \in U$ in the direction of $\xi \in D(U)$ is

$$\begin{aligned} \left\langle \frac{\delta S(\psi, c, e)}{\delta \psi}, \xi \right\rangle_{D'(U), D(U)} &= \left\langle \frac{\partial s(\psi, c, e)}{\partial \psi}, \xi \right\rangle_{D'(U), D(U)} \\ &\quad - \left\langle \frac{\partial}{\partial \psi} \left(\frac{\epsilon_{\theta}^2}{2} |\nabla \psi|^2 \right), \xi \right\rangle_{D'(U), D(U)}, \end{aligned} \quad (1.7)$$

where $D'(U)$ is the space of distributions corresponding to the space $D(U)$ of test functions on U with compact support.

Consider now the term

$$\mathcal{I} = \left\langle \frac{\partial}{\partial \psi} \left(\frac{\epsilon_{\theta}^2}{2} |\nabla \psi|^2 \right), \xi \right\rangle_{D'(U), D(U)} = \int_{\Omega} \frac{\partial}{\partial \psi} \left(\frac{\epsilon_{\theta}^2}{2} |\nabla \psi|^2 \right) \xi d\mathbf{x}$$

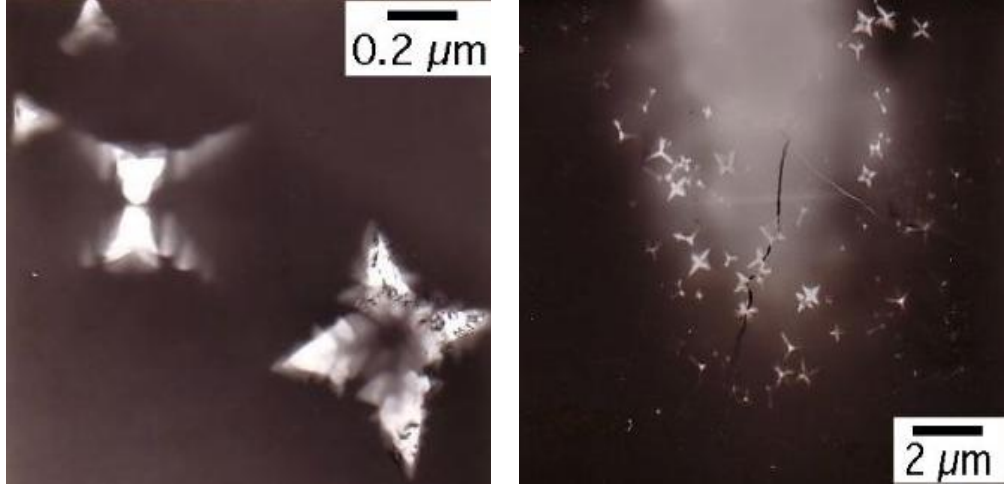


Figure 1.2: Early stages of dendrites of the Fe-Si during the solidification provided by Andrew Fairbank (University of Wollongong Australia)

and carrying out the differentiation in the integrand on the right hand side of the above equation with respect to ψ , we have

$$\mathcal{I} = \int_{\Omega} \left(\epsilon_{\theta} |\nabla \psi|^2 \frac{\partial \epsilon_{\theta}}{\partial \psi} \cdot \xi + \epsilon_{\theta}^2 \nabla \psi \cdot \nabla \xi \right) d\mathbf{x}.$$

Since ϵ_{θ} is a function of θ , therefore applying the chain rule and using divergence theorem, we arrive at

$$\mathcal{I} = \int_{\Omega} \left(\epsilon_{\theta} |\nabla \psi|^2 \frac{\partial \epsilon_{\theta}}{\partial \theta} \frac{\partial \theta}{\partial \psi} \cdot \xi - \text{div} (\epsilon_{\theta}^2 \nabla \psi) \xi \right) d\mathbf{x}. \quad (1.8)$$

Therefore the variational derivative of S can be given as

$$\frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \text{div} (\epsilon_{\theta}^2 \nabla \psi) - A \left(\epsilon_{\theta}, \epsilon'_{\theta}, \frac{\partial \theta}{\partial \psi}, \nabla \psi \right), \quad (1.9)$$

where $\epsilon'_{\theta} = \partial \epsilon_{\theta} / \partial \theta$ and $A \left(\epsilon_{\theta}, \epsilon'_{\theta}, \frac{\partial \theta}{\partial \psi}, \nabla \psi \right) = \epsilon_{\theta} \epsilon'_{\theta} \frac{\partial \theta}{\partial \psi} |\nabla \psi|^2$.

Now we shall compute the derivative of the entropy density $s(\psi, c, e)$ with respect to ψ , i.e. $\partial s / \partial \psi$ using free energy density $f(\psi, c, T)$.

As we know from the basic thermodynamics that the free energy density can be defined by

$$\left. \begin{aligned} f(\psi, c, T) &= e(\psi, c, T) - Ts(\psi, c, e), \\ \frac{1}{T} &= \frac{\partial s}{\partial e}(\psi, c, e), \end{aligned} \right\} \quad (1.10)$$

where $e(\psi, c, T)$ and $s(\psi, c, T)$ are the internal energy density and entropy density of the binary alloy and $T(\mathbf{x}, t)$ is the temperature at any point in the time-space domain. Taking differential of the above equation we have

$$df(\psi, c, T) = de(\psi, c, T) - Tds(\psi, c, T) - s(\psi, c, T)dT,$$

or

$$df(\psi, c, T) = de(\psi, c, T) - T \left(\frac{\partial s}{\partial e} de + \frac{\partial s}{\partial \psi} d\psi + \frac{\partial s}{\partial c} dc \right) - s dT.$$

Using the definition of the temperature (*i.e.*, $1/T = \partial s / \partial e$), the above equation takes the form

$$df(\psi, c, T) = -T \frac{\partial s}{\partial \psi} d\psi - T \frac{\partial s}{\partial c} dc - s dT. \quad (1.11)$$

Also as we know that

$$df(\psi, c, T) = \frac{\partial f}{\partial \psi} d\psi + \frac{\partial f}{\partial c} dc + \frac{\partial f}{\partial T} dT. \quad (1.12)$$

Comparing equation (1.11) and (1.12), we have the following relations

$$\frac{\partial s(\psi, c, T)}{\partial \psi} = -\frac{1}{T} \frac{\partial f(\psi, c, T)}{\partial \psi}, \quad (1.13)$$

$$\frac{\partial s(\psi, c, T)}{\partial c} = -\frac{1}{T} \frac{\partial f(\psi, c, T)}{\partial c}, \quad (1.14)$$

$$\frac{\partial f(\psi, c, T)}{\partial T} = -s(\psi, c, T). \quad (1.15)$$

An explicit relation of the free energy density $f(\psi, c, T)$ of a binary alloy is given in [24] as

$$f(\psi, c, T) = (1 - c)\mu_A(\psi, c, T) + c\mu_B(\psi, c, T), \quad (1.16)$$

where $\mu_A(\psi, c, T)$ and $\mu_B(\psi, c, T)$ are the corresponding chemical potentials of the two constituent species, and are defined by

$$\mu_A(\psi, c, T) = f_A(\psi, T) + \lambda(\psi)c^2 + \frac{RT}{V_m} \ln(1 - c), \quad (1.17)$$

$$\mu_B(\psi, c, T) = f_B(\psi, T) + \lambda(\psi)(1 - c)^2 + \frac{RT}{V_m} \ln(c), \quad (1.18)$$

where $f_A(\psi, T)$ and $f_B(\psi, T)$ are the free energy densities for substances A and B respectively, R is the universal gas constant, V_m is the molar volume and $\lambda(\psi)$ the regular solution interaction parameter associated with the enthalpy of mixing and is assumed to be

$$\lambda(\psi) = \lambda_S + p(\psi)(\lambda_L - \lambda_S),$$

where the parameters λ_S and λ_L are the enthalpies of mixing of the solid and liquid respectively.

Here it is assumed that the solution is ideal [24], therefore the parameters λ_S and λ_L are assumed to be zero and hence $\lambda(\psi) = 0$.

Now using the basic thermodynamic, the relationship for the free energy density of the pure substance can be given as

$$f_I(\psi, T) = e_I(\psi, T) - Ts_I(\psi, T), \quad I = A, B. \quad (1.19)$$

where $e_I(\psi, T)$ is the internal energy density and $s_I(\psi, T)$ is the entropy density of the pure substance I where $I = A, B$.

The internal energy density for each substance is assumed to have the form in [24] as

$$e_I(\psi, T) = e_{I,S}(T) + p(\psi)(e_{I,L}(T) - e_{I,S}(T)), \quad I = A, B. \quad (1.20)$$

where $e_{I,S}(T)$ and $e_{I,L}(T)$ are the solid and liquid internal energies of the pure substances I where $I = A, B$, and are further defined as

$$e_{I,S}(T) = e_{I,S}(T_m^I) + C_S^I(T - T_m^I), \quad (1.21)$$

$$e_{I,L}(T) = e_{I,L}(T_m^I) + C_L^I(T - T_m^I), \quad (1.22)$$

where T_m^I is the melting temperature, C_S^I and C_L^I are the heat capacities of solid and liquid and $e_{I,S}(T_m^I)$ and $e_{I,L}(T_m^I)$ are the internal energies of solid and liquid at the melting temperature respectively of the substance I , where $I = A, B$.

The factor $p(\psi)$ should be selected here in the way that it is 0 in the solid phase to recover the internal energy density of solid and 1 in the liquid phase to obtain the internal energy density of the liquid for the pure substance I , that is, it should satisfy the following conditions

$$\begin{aligned} p(0) &= p(1) = 0 \\ p'(\psi) &> 0, \quad \forall \psi \in]0, 1[. \end{aligned} \quad (1.23)$$

We shall elucidate further the choice of $p(\psi)$ below.

The latent heat of each pure substance is defined as

$$L_I = e_{I,L}(T_m^I) - e_{I,S}(T_m^I), \quad I = A, B. \quad (1.24)$$

Supposing that heat capacities are identical (*i.e.* $C_S^I = C_L^I = C_I$) for solid and liquid phase of each substance, we can write the final form of the internal energy densities of each substance using equations (1.21), (1.22) and (1.24) as

$$e_I(\psi, T) = e_{I,S}(T_m^I) + C_I(T - T_m^I) + p(\psi)L_I, \quad I = A, B. \quad (1.25)$$

Now using equation (1.15), the equation (1.19) can be written as

$$f_I(\psi, T) = e_I(\psi, T) + T \frac{\partial f_I}{\partial T}(\psi, T), \quad I = A, B.$$

or

$$T \frac{\partial f_I}{\partial T}(\psi, T) - f_I(\psi, T) + e_I(\psi, T) = 0, \quad I = A, B.$$

The above equation can further be written as

$$T^2 \frac{\partial (f_I/T)}{\partial T} + e_I(\psi, T) = 0$$

and dividing by T^2 , we get

$$\frac{\partial (f_I/T)}{\partial T} + \frac{e_I(\psi, T)}{T^2} = 0.$$

Integrating above equation with respect to T from T to T_m^I , we have

$$\frac{1}{T_m^I} f_I(\psi, T_m^I) - \frac{1}{T} f_I(\psi, T) + \int_T^{T_m^I} \frac{e_I(\psi, \tau)}{\tau^2} d\tau = 0,$$

or

$$f_I(\psi, T) = T \left(\int_T^{T_m^I} \frac{e_I(\psi, \tau)}{\tau^2} d\tau + \frac{1}{T_m^I} f_I(\psi, T_m^I) \right).$$

According to equation (1.25), we have

$$f_I(\psi, T) = T \left(\int_T^{T_m^I} \frac{e_{I,S}(T_m^I) + C_I(\tau - T_m^I) + p(\psi)L_I}{\tau^2} d\tau + \frac{1}{T_m^I} f_I(\psi, T_m^I) \right).$$

Simplifying above equation, we arrive at

$$\begin{aligned} f_I(\psi, T) = \frac{T}{T_m^I} f_I(\psi, T_m^I) + e_{I,S}(T_m^I) \left(1 - \frac{T}{T_m^I} \right) + C_I T_m^I \left(\frac{T}{T_m^I} - 1 \right) \\ + L_I p(\psi) \left(1 - \frac{T}{T_m^I} \right) + C_I T \ln \left(\frac{T_m^I}{T} \right), \end{aligned}$$

or

$$\begin{aligned} f_I(\psi, T) = \frac{T}{T_m^I} f_I(\psi, T_m^I) + (e_{I,S}(T_m^I) - C_I T_m^I + L_I p(\psi)) \left(1 - \frac{T}{T_m^I} \right) \\ - C_I T \ln \left(\frac{T}{T_m^I} \right). \end{aligned} \quad (1.26)$$

Now the expression $f_I(\psi, T_m^I)$ is left only to be determined to achieve the final form of the free energy density of each substance. The choice of $f_I(\psi, T_m^I)$ is dependent on

the phase field variable ψ as we should have the free energy density of the substance I in the solid phase at $\psi = 0$ and in the liquid phase at $\psi = 1$. Also the free energy density should be symmetric at the melting temperature with respect to $\psi = 1/2$. Thus the free energy density $f_I(\psi, T_m^I)$ that follow these conditions can be chosen as a function $g(\psi)$ of class $C^2([0, 1], R)$ which satisfy the following conditions

$$\left. \begin{aligned} g(0) &= g(1) = 0, \\ g'(\psi) &= 0 \text{ iff } \psi \in \{0, 1/2, 1\}, \\ g''(0), g''(1) &> 0, \\ g(\psi) &= g(1 - \psi). \end{aligned} \right\} \quad (1.27)$$

This function is being chosen in [24] as

$$g(\psi) = \psi^2(1 - \psi)^2, \quad (1.28)$$

which is a double well polynomial function of the minimum degree that satisfy the properties defined in equation (1.27). More details about the choice and properties of the function $g(\psi)$ can be found in [24], [14]. Therefore the form of $f_I(\psi, T_m^I)$ is assumed to be

$$f_I(\psi, T_m^I) = T_m^I W_I \psi^2(1 - \psi)^2, \quad (1.29)$$

where W_I is the constant which control the height of the well and is defined as

$$W_I = \frac{3\sigma_I}{\sqrt{2}T_m^I\delta_I}, \quad I = A, B. \quad (1.30)$$

where σ_I is the solid-liquid interface energy, T_m^I is the melting temperature and δ_I is the interface thickness of the pure substance I . The graph of the $f_I(\psi, T_m^I)$ is given in the Fig. 1.3. Note that, to show that the minima of $f_I(\psi, T_m^I)$ lie only in the interval $[0, 1]$, we have taken the domain interval as $[-0.5, 1.5]$ for $f_I(\psi, T_m^I)$ in the figure.

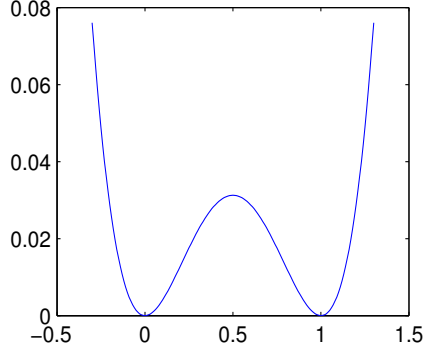
Now we shall determine an expression for $p(\psi)$ by demanding that the only stable states of the system are the solid and liquid states and there are only two minima of the free energy density $f_I(\psi, T)$ at $\psi = 0$ and $\psi = 1$ for any temperature $T(\mathbf{x}, t)$.

Differentiating equation (1.26) with respect to ψ and using equation (1.29), we have

$$\frac{\partial f_I(\psi, T)}{\partial \psi} = W_I T g'(\psi) + L_I p'(\psi) \left(1 - \frac{T}{T_m^I}\right),$$

where $g'(\psi) = \partial g(\psi)/\partial \psi$ and $p'(\psi) = \partial p(\psi)/\partial \psi$.

As $g'(0) = g'(1) = 0$, we note from the above equation that $\partial f_I/\partial \psi$ is zero at $\psi = 0$ and $\psi = 1$ only if $p'(0) = p'(1) = 0$ for any temperature $T(\mathbf{x}, t)$. To ensure that

Figure 1.3: The graph of $f_I(\psi, T_m^I)$

the only minima of the free energy density $f_I(\psi, T)$ are at $\psi = 0$ and $\psi = 1$ for any temperature, the function $p(\psi)$ is required to fulfill the following conditions along with the conditions defined earlier and that it is of the class $C^2([0, 1], R)$

$$\left. \begin{aligned} p(0) &= 0, \quad p(1) = 1, \\ p'(0) &= p'(1) = 0, \\ p''(0) &= p''(1) = 0, \\ p'(\psi) &> 0, \quad \forall \psi \in (0, 1). \end{aligned} \right\} \quad (1.31)$$

Here it is chosen to have the form [24]

$$p(\psi) = \psi^3 (10 - 15\psi + 6\psi^2), \quad (1.32)$$

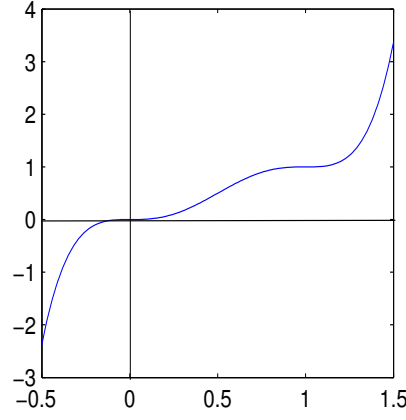
which satisfy the conditions defined in equation (1.31). More details about the choice and properties of the function $p(\psi)$ can be found in [14], [46]. The graph of the function $p(\psi)$ is given in the Fig. 1.4. Note that, to show the behavior of the function $p(\psi)$ well within the interval $[0, 1]$, we have taken the domain interval as $[-0.5, 1.5]$.

Thus the final form of the free energy density for the substance I , where $I = A, B$, can be given as

$$\begin{aligned} f_I(\psi, T) &= W_I T g(\psi) + (e_{I,s}(T_m^I) - C_I T_m^I + L_I p(\psi)) \left(1 - \frac{T}{T_m^I}\right) \\ &\quad - C_I T \ln \left(\frac{T}{T_m^I}\right). \end{aligned} \quad (1.33)$$

Now using equations (1.16)-(1.18) in the equation (1.13), we have

$$\begin{aligned} \frac{\partial s}{\partial \psi} &= -\frac{1}{T} \frac{\partial}{\partial \psi} \left\{ (1-c) \left(f_A(\psi, T) + \frac{RT}{V_m} \ln(1-c) \right) \right\} \\ &\quad + \frac{1}{T} \frac{\partial}{\partial \psi} \left\{ c \left(f_B(\psi, T) + \frac{RT}{V_m} \ln(c) \right) \right\}, \end{aligned}$$

Figure 1.4: The graph of function $p(\psi)$.

where $\lambda(\psi) = 0$.

Making use of equation (1.33) and carrying out the differentiation with respect to ψ , the above equation becomes

$$\begin{aligned} \frac{\partial s}{\partial \psi} = & -\frac{1}{T}(1-c) \left(W_A g'(\psi) T + p'(\psi) L_A \left(1 - \frac{T}{T_m^A} \right) \right) \\ & - \frac{1}{T} c \left(W_B g'(\psi) T + p'(\psi) L_B \left(1 - \frac{T}{T_m^B} \right) \right). \end{aligned}$$

As $p'(\psi) = 30g(\psi)$, thus we have

$$\begin{aligned} \frac{\partial s}{\partial \psi} = & -(1-c) \left\{ W_A g'(\psi) + 30g(\psi) L_A \left(\frac{1}{T} - \frac{1}{T_m^A} \right) \right\} \\ & - c \left\{ W_B g'(\psi) T + 30g(\psi) L_B \left(\frac{1}{T} - \frac{1}{T_m^B} \right) \right\}. \end{aligned}$$

The above equation can be written as

$$\frac{\partial s}{\partial \psi} = -(1-c)H_A(\psi, T) - cH_B(\psi, T), \quad (1.34)$$

where

$$H_A(\psi, T) = W_A g'(\psi) + 30g(\psi) L_A \left(\frac{1}{T} - \frac{1}{T_m^A} \right), \quad (1.35)$$

$$H_B(\psi, T) = W_B g'(\psi) + 30g(\psi) L_B \left(\frac{1}{T} - \frac{1}{T_m^B} \right), \quad (1.36)$$

with $g'(\psi) = \partial g(\psi)/\partial \psi$.

Substituting equation (1.34) into the equation (1.9) and then the resulting equation in the equation (1.3), we obtain the following equation

$$\frac{D\psi}{Dt} = M_\psi \left(\operatorname{div} (\epsilon_\theta^2 \nabla \psi) - (1-c)H_A(\psi, T) - cH_B(\psi, T) - A\left(\epsilon_\theta, \epsilon'_\theta, \frac{\partial \theta}{\partial \psi}, \nabla \psi\right) \right), \quad (1.37)$$

which is the general equation of phase-field, where the operators A and $\operatorname{div} (\epsilon_\theta^2 \nabla \psi)$ are left to be calculated. We can compute these operators by introducing the operator ϵ_θ .

If we assume that the interface thickness $\delta_A = \delta_B = \delta$ in the constants defined in equation (1.30) and that the solidification process is isotropic (i.e., $\epsilon_\theta = \epsilon_0$ is a constant), then the equation (1.37) simplifies and takes the form as

$$\frac{D\psi}{Dt} = M_\psi \epsilon_0^2 \left(\Delta \psi - \frac{\lambda_1(c)}{\delta^2} g'(\psi) - \frac{1}{\delta} \lambda_2(c) p'(\psi) \right), \quad (1.38)$$

where

$$\left. \begin{aligned} \epsilon_0^2 &= 3\sqrt{2}(\sigma_A + \sigma_B)\delta/T_m, \\ T_m &= (T_m^A + T_m^B)/2, \\ \lambda_1(c) &= (1-c)\lambda_{1A} + c\lambda_{1B}, \\ \lambda_2(c) &= (1-c)\lambda_{2A} + c\lambda_{2B}, \end{aligned} \right\} \quad (1.39)$$

with

$$\begin{aligned} \lambda_{1A} &= \frac{\sigma_A}{(\sigma_A + \sigma_B)} \frac{T_m}{T_m^A}, & \lambda_{1B} &= \frac{\sigma_B}{(\sigma_A + \sigma_B)} \frac{T_m}{T_m^B}, \\ \lambda_{2A} &= \frac{L_A T_m}{3\sqrt{2}(\sigma_A + \sigma_B)} \left(\frac{1}{T} - \frac{1}{T_m^A} \right), & \lambda_{2B} &= \frac{L_B T_m}{3\sqrt{2}(\sigma_A + \sigma_B)} \left(\frac{1}{T} - \frac{1}{T_m^B} \right). \end{aligned}$$

1.3.1 Two dimensional case

In two dimensions, the parameter ϵ_θ is assumed to be anisotropic and is defined as [24]

$$\epsilon_\theta = \epsilon_0 \eta = \epsilon_0 (1 + \gamma_0 \cos k\theta), \quad (1.40)$$

where anisotropic means that ϵ_θ is dependent on the direction of the solid-liquid interface, γ_0 is the anisotropic amplitude, k the mode number, ϵ_0 is a constant and

$$\theta = \arctan \left(\frac{\psi_y}{\psi_x} \right), \quad (1.41)$$

is the angle between the local interface normal and a designated base vector of the crystal lattice, subscripts x and y are used to denote the partial derivatives with respect to spatial coordinates, that is, $\psi_x = \partial\psi/\partial x$ and $\psi_y = \partial\psi/\partial y$.

The anisotropy plays an important role in modeling the dendritic solidification process. In fact, for example, for the metal alloys, the form of dendrites is usually symmetric and has four major dendrite arms and minor arms around them (see e.g., Fig. 1.2). In the solidification model, the mode number k , in the anisotropy function ϵ_θ , usually represent the dendrite arms. If we want to obtain a dendrite with four arms, we fix the value of k equal to 4. Its value depends on the form of dendrites obtained in a particular alloy.

To compute the operators A and $\text{div}(\epsilon_\theta^2 \nabla \psi)$, we shall restart from the equation (1.8) given by

$$\mathcal{I} = \int_{\Omega} \left(\epsilon_\theta |\nabla \psi|^2 \frac{\partial \epsilon_\theta}{\partial \theta} \frac{\partial \theta}{\partial \psi} \cdot \xi + \text{div}(\epsilon_\theta^2 \nabla \psi) \xi \right) d\mathbf{x}. \quad (1.42)$$

Now taking derivative of equation (1.41) with respect to ψ in the direction of ξ , we get

$$\begin{aligned} \frac{\partial \theta}{\partial \psi} \cdot \xi &= \frac{\partial}{\partial \psi} \left(\arctan \left(\frac{\psi_y}{\psi_x} \right) \right) \cdot \xi, \\ \frac{\partial \theta}{\partial \psi} \cdot \xi &= \frac{1}{1 + \left(\frac{\psi_y}{\psi_x} \right)^2} \left(\frac{\psi_x \xi_y - \psi_y \xi_x}{\psi_x^2} \right), \\ \frac{\partial \theta}{\partial \psi} \cdot \xi &= \frac{\psi_x \xi_y - \psi_y \xi_x}{|\nabla \psi|^2}. \end{aligned} \quad (1.43)$$

Substituting equation (1.43) in the equation (1.42), we arrive at

$$\mathcal{I} = \int_{\Omega} \left(\epsilon_\theta \epsilon'_\theta \left(\frac{\psi_x \xi_y - \psi_y \xi_x}{|\nabla \psi|^2} \right) |\nabla \psi|^2 - \epsilon_\theta^2 \nabla \psi \cdot \nabla \xi \right) d\mathbf{x},$$

where $\epsilon'_\theta = \partial \epsilon_\theta / \partial \theta$.

Simplifying above equation, we have

$$\mathcal{I} = \int_{\Omega} (\epsilon_\theta \epsilon'_\theta \psi_x \xi_y - \epsilon_\theta \epsilon'_\theta \psi_y \xi_x) d\mathbf{x} - \int_{\Omega} \epsilon_\theta^2 (\psi_x \xi_x + \psi_y \xi_y) d\mathbf{x}.$$

The above equation can further be written as

$$\begin{aligned} \mathcal{I} &= \int_{\Omega} \left(\frac{\partial}{\partial y} (\epsilon_\theta \epsilon'_\theta \psi_x \xi) - \xi \frac{\partial}{\partial y} (\epsilon_\theta \epsilon'_\theta \psi_x) - \frac{\partial}{\partial x} (\epsilon_\theta \epsilon'_\theta \psi_y \xi) + \xi \frac{\partial}{\partial x} (\epsilon_\theta \epsilon'_\theta \psi_y) \right) d\mathbf{x} \\ &\quad - \int_{\Omega} \left(\frac{\partial}{\partial x} (\epsilon_\theta^2 \psi_x \xi) - \xi \frac{\partial}{\partial x} (\epsilon_\theta^2 \psi_x) + \frac{\partial}{\partial y} (\epsilon_\theta^2 \psi_y \xi) - \xi \frac{\partial}{\partial y} (\epsilon_\theta^2 \psi_y) \right) d\mathbf{x}. \end{aligned}$$

Simplification of above equation yields

$$\begin{aligned} \mathcal{I} = & \int_{\Omega} \left(\frac{\partial}{\partial x} (\epsilon_{\theta} \epsilon'_{\theta} \psi_y) - \frac{\partial}{\partial y} (\epsilon_{\theta} \epsilon'_{\theta} \psi_x) - \left(\frac{\partial}{\partial x} (\epsilon_{\theta}^2 \psi_x) + \frac{\partial}{\partial y} (\epsilon_{\theta}^2 \psi_y) \right) \right) \xi d\mathbf{x} \\ & + \int_{\Omega} \left(\frac{\partial}{\partial y} (\epsilon_{\theta} \epsilon'_{\theta} \psi_x \xi + \epsilon_{\theta}^2 \psi_y \xi) - \frac{\partial}{\partial x} (\epsilon_{\theta} \epsilon'_{\theta} \psi_y \xi - \epsilon_{\theta}^2 \psi_x \xi) \right) d\mathbf{x}, \end{aligned}$$

or

$$\begin{aligned} \mathcal{I} = & \int_{\Omega} \left(\frac{\partial}{\partial x} (\epsilon_{\theta} \epsilon'_{\theta} \psi_y) - \frac{\partial}{\partial y} (\epsilon_{\theta} \epsilon'_{\theta} \psi_x) - \nabla \cdot (\epsilon_{\theta}^2 \nabla \psi) \right) \xi d\mathbf{x} \\ & + \int_{\Omega} \left(\frac{\partial}{\partial y} ((\epsilon_{\theta} \epsilon'_{\theta} \psi_x + \epsilon_{\theta}^2 \psi_y) \xi) - \frac{\partial}{\partial x} ((\epsilon_{\theta} \epsilon'_{\theta} \psi_y - \epsilon_{\theta}^2 \psi_x) \xi) \right) d\mathbf{x}. \end{aligned}$$

Since $\xi = 0$ on the boundary, therefore the second term in the above equation vanishes

$$\mathcal{I} = \int_{\Omega} \left(\frac{\partial}{\partial x} (\epsilon_{\theta} \epsilon'_{\theta} \psi_y) - \frac{\partial}{\partial y} (\epsilon_{\theta} \epsilon'_{\theta} \psi_x) - \nabla \cdot (\epsilon_{\theta}^2 \nabla \psi) \right) \xi d\mathbf{x}. \quad (1.44)$$

According to equation (1.44), the equation (1.7) takes the form

$$\begin{aligned} \left\langle \frac{\delta S}{\delta \psi}, \xi \right\rangle_{D'(U), D(U)} &= \left\langle \frac{\partial s}{\partial \psi}, \xi \right\rangle_{D'(U), D(U)} \\ &- \int_{\Omega} \left(\frac{\partial}{\partial x} (\epsilon_{\theta} \epsilon'_{\theta} \psi_y) - \frac{\partial}{\partial y} (\epsilon_{\theta} \epsilon'_{\theta} \psi_x) - \nabla \cdot (\epsilon_{\theta}^2 \nabla \psi) \right) \xi d\mathbf{x}. \end{aligned}$$

Thus we have the variational derivative of equation (1.6) as

$$\frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} - \frac{\partial}{\partial x} (\epsilon_{\theta} \epsilon'_{\theta} \psi_y) + \frac{\partial}{\partial y} (\epsilon_{\theta} \epsilon'_{\theta} \psi_x) + \nabla \cdot (\epsilon_{\theta}^2 \nabla \psi).$$

As $\epsilon_{\theta} = \epsilon_0 \eta$, the above equation becomes

$$\frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \nabla \cdot (\eta^2 \nabla \psi) - \epsilon_0^2 \frac{\partial}{\partial x} (\eta \eta' \psi_y) + \epsilon_0^2 \frac{\partial}{\partial y} (\eta \eta' \psi_x), \quad (1.45)$$

where $\eta' = \partial \eta / \partial \theta$ and ϵ_0 is a constant.

Carrying out the partial derivatives of x and y in the above equation, we have

$$\begin{aligned} \frac{\delta S}{\delta \psi} = & \frac{\partial s}{\partial \psi} + \epsilon_0^2 \nabla \cdot (\eta^2 \nabla \psi) - \epsilon_0^2 \left(\eta \eta' \psi_{xy} + \eta \psi_y \frac{\partial \eta'}{\partial x} + \eta' \psi_y \frac{\partial \eta}{\partial x} \right) \\ & + \epsilon_0^2 \left(\eta \eta' \psi_{yx} + \eta \psi_x \frac{\partial \eta'}{\partial y} + \eta' \psi_x \frac{\partial \eta}{\partial y} \right). \end{aligned}$$

Since $\psi_{xy} = \psi_{yx}$, therefore the above equation simplifies as

$$\begin{aligned} \frac{\delta S}{\delta \psi} = & \frac{\partial s}{\partial \psi} + \epsilon_0^2 \nabla \cdot (\eta^2 \nabla \psi) - \epsilon_0^2 \left(\eta \psi_y \frac{\partial \eta'}{\partial x} + \eta' \psi_y \frac{\partial \eta}{\partial x} \right) \\ & + \epsilon_0^2 \left(\eta \psi_x \frac{\partial \eta'}{\partial y} + \eta' \psi_x \frac{\partial \eta}{\partial y} \right). \end{aligned}$$

As η is function of θ , therefore applying chain rule, we have

$$\begin{aligned} \frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \nabla \cdot (\eta^2 \nabla \psi) - \epsilon_0^2 \left(\eta \psi_y \frac{\partial \eta'}{\partial \theta} \frac{\partial \theta}{\partial x} + \eta' \psi_y \frac{\partial \eta}{\partial \theta} \frac{\partial \theta}{\partial x} \right) \\ + \epsilon_0^2 \left(\eta \psi_x \frac{\partial \eta'}{\partial \theta} \frac{\partial \theta}{\partial y} + \eta' \psi_x \frac{\partial \eta}{\partial \theta} \frac{\partial \theta}{\partial y} \right). \end{aligned}$$

Differentiating equation (1.41) with respect to x and y respectively, we get

$$\frac{\partial \theta}{\partial x} = \frac{\psi_x \psi_{xy} - \psi_y \psi_{xx}}{|\nabla \psi|^2}, \quad \frac{\partial \theta}{\partial y} = \frac{\psi_x \psi_{yy} - \psi_y \psi_{yx}}{|\nabla \psi|^2}. \quad (1.46)$$

Making use of equation (1.46) in the previous equation, we have

$$\begin{aligned} \frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \nabla \cdot (\eta^2 \nabla \psi) - \epsilon_0^2 \eta \eta'' \psi_y \left(\frac{\psi_x \psi_{xy} - \psi_y \psi_{xx}}{|\nabla \psi|^2} \right) \\ - \epsilon_0^2 (\eta')^2 \psi_y \left(\frac{\psi_x \psi_{xy} - \psi_y \psi_{xx}}{|\nabla \psi|^2} \right) + \epsilon_0^2 \eta \eta'' \psi_x \left(\frac{\psi_x \psi_{yy} - \psi_y \psi_{yx}}{|\nabla \psi|^2} \right) \\ + \epsilon_0^2 (\eta')^2 \psi_x \left(\frac{\psi_x \psi_{yy} - \psi_y \psi_{yx}}{|\nabla \psi|^2} \right), \end{aligned}$$

or

$$\begin{aligned} \frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \nabla \cdot (\eta^2 \nabla \psi) - \epsilon_0^2 \eta \eta'' \left(\frac{\psi_x \psi_y \psi_{xy} - \psi_y^2 \psi_{xx}}{|\nabla \psi|^2} \right) \\ - \epsilon_0^2 (\eta')^2 \left(\frac{\psi_x \psi_y \psi_{xy} - \psi_y^2 \psi_{xx}}{|\nabla \psi|^2} \right) + \epsilon_0^2 \eta \eta'' \left(\frac{\psi_x^2 \psi_{yy} - \psi_x \psi_y \psi_{yx}}{|\nabla \psi|^2} \right) \\ + \epsilon_0^2 (\eta')^2 \left(\frac{\psi_x^2 \psi_{yy} - \psi_x \psi_y \psi_{yx}}{|\nabla \psi|^2} \right). \end{aligned} \quad (1.47)$$

Now consider the following term

$$\mathcal{I}_1 = \nabla \cdot (\eta^2 \nabla \psi).$$

By employing the Green's identity $(\nabla \cdot (f_1 \nabla f_2) = f_1 \Delta f_2 + \nabla f_1 \cdot \nabla f_2)$, we have

$$\mathcal{I}_1 = \eta^2 \Delta \psi + \nabla \eta^2 \cdot \nabla \psi.$$

The above equation can be written as

$$\mathcal{I}_1 = \eta^2 \Delta \psi + 2\eta \nabla \eta \cdot \nabla \psi.$$

Since η is the function of θ , therefore by using chain rule we arrive at

$$\mathcal{I}_1 = \eta^2 \Delta \psi + 2\eta \eta' \nabla \theta \cdot \nabla \psi.$$

According to equation (1.46), the above equations becomes

$$\mathcal{I}_1 = \eta^2 \Delta \psi + \frac{2\eta\eta'}{|\nabla\psi|^2} (\psi_x (\psi_x \psi_{xy} - \psi_y \psi_{xx}) + \psi_y (\psi_x \psi_{yy} - \psi_y \psi_{yx})). \quad (1.48)$$

Making use of equation (1.48) in the equation (1.47), we have

$$\begin{aligned} \frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \left(\eta^2 \Delta \psi + \frac{2\eta\eta'}{|\nabla\psi|^2} (\psi_x (\psi_x \psi_{xy} - \psi_y \psi_{xx}) + \psi_y (\psi_x \psi_{yy} - \psi_y \psi_{yx})) \right) \\ - \epsilon_0^2 \eta \eta'' \left(\frac{\psi_x \psi_y \psi_{xy} - \psi_y^2 \psi_{xx}}{|\nabla\psi|^2} \right) - \epsilon_0^2 (\eta')^2 \left(\frac{\psi_x \psi_y \psi_{xy} - \psi_y^2 \psi_{xx}}{|\nabla\psi|^2} \right) \\ + \epsilon_0^2 \eta \eta'' \left(\frac{\psi_x^2 \psi_{yy} - \psi_x \psi_y \psi_{yx}}{|\nabla\psi|^2} \right) + \epsilon_0^2 (\eta')^2 \left(\frac{\psi_x^2 \psi_{yy} - \psi_x \psi_y \psi_{yx}}{|\nabla\psi|^2} \right). \end{aligned}$$

Simplification of the above equation yields

$$\begin{aligned} \frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \eta^2 \Delta \psi + \epsilon_0^2 \eta \eta' \left(\frac{2\psi_x \psi_y}{|\nabla\psi|^2} (\psi_{yy} - \psi_{xx}) + \frac{2\psi_{xy}}{|\nabla\psi|^2} (\psi_x^2 - \psi_y^2) \right) \\ + \frac{\epsilon_0^2 \eta \eta''}{|\nabla\psi|^2} (-2\psi_x \psi_y \psi_{xy} + \psi_y^2 \psi_{xx} + \psi_x^2 \psi_{yy}) \\ + \frac{\epsilon_0^2 (\eta')^2}{|\nabla\psi|^2} (-2\psi_x \psi_y \psi_{xy} + \psi_y^2 \psi_{xx} + \psi_x^2 \psi_{yy}), \end{aligned}$$

or

$$\begin{aligned} \frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \eta^2 \Delta \psi + \epsilon_0^2 \eta \eta' \left(\frac{2\psi_x \psi_y}{|\nabla\psi|^2} (\psi_{yy} - \psi_{xx}) + \frac{2\psi_{xy}}{|\nabla\psi|^2} (\psi_x^2 - \psi_y^2) \right) \\ + \frac{\epsilon_0^2 (\eta \eta'' + (\eta')^2)}{|\nabla\psi|^2} (-2\psi_x \psi_y \psi_{xy} + \psi_y^2 \psi_{xx} + \psi_x^2 \psi_{yy}). \end{aligned}$$

Using equation (1.41), we can easily have

$$\begin{aligned} \cos\theta &= \frac{\psi_x}{|\nabla\psi|}, \quad \sin\theta = \frac{\psi_y}{|\nabla\psi|}, \\ \cos 2\theta &= \frac{\psi_x^2 - \psi_y^2}{|\nabla\psi|^2}, \quad \sin 2\theta = \frac{2\psi_x \psi_y}{|\nabla\psi|^2}. \end{aligned} \quad (1.49)$$

With the help of relations (1.49) and multiplying and dividing the last term of the above equation by 2 and rearranging the terms, we obtain

$$\begin{aligned} \frac{\delta S}{\delta \psi} = \frac{\partial s}{\partial \psi} + \epsilon_0^2 \eta^2 \Delta \psi + \epsilon_0^2 \eta \eta' (\sin 2\theta (\psi_{yy} - \psi_{xx}) + 2\psi_{xy} \cos 2\theta) \\ + \frac{\epsilon_0^2 (\eta \eta'' + (\eta')^2)}{2} \left(-\frac{4\psi_x \psi_y \psi_{xy}}{|\nabla\psi|^2} + \frac{\psi_y^2 \psi_{xx} + \psi_y^2 \psi_{xx} + \psi_x^2 \psi_{yy} + \psi_x^2 \psi_{yy}}{|\nabla\psi|^2} \right). \end{aligned}$$

Adding and subtracting $\psi_x^2\psi_{xx}$ and $\psi_y^2\psi_{yy}$ in the last term of the above equation and then factorizing the terms, we arrive at

$$\begin{aligned} \frac{\delta S}{\delta \psi} = & \frac{\partial s}{\partial \psi} + \epsilon_0^2 \eta^2 \Delta \psi + \epsilon_0^2 \eta \eta' (\sin 2\theta (\psi_{yy} - \psi_{xx}) + 2\psi_{xy} \cos 2\theta) \\ & - \frac{\epsilon_0^2 (\eta \eta'' + (\eta')^2)}{2} \left(\frac{4\psi_x \psi_y \psi_{xy}}{|\nabla \psi|^2} - (\psi_{xx} + \psi_{yy}) - \frac{(\psi_x^2 - \psi_y^2)(\psi_{yy} - \psi_{xx})}{|\nabla \psi|^2} \right), \end{aligned}$$

According to equation (1.49), the above equation takes the form

$$\begin{aligned} \frac{\delta S}{\delta \psi} = & \frac{\partial s}{\partial \psi} + \epsilon_0^2 \eta^2 \Delta \psi + \epsilon_0^2 \eta \eta' (\sin 2\theta (\psi_{yy} - \psi_{xx}) + 2\psi_{xy} \cos 2\theta) \\ & - \frac{\epsilon_0^2 (\eta \eta'' + (\eta')^2)}{2} (2\psi_{xy} \sin 2\theta - \Delta \psi - (\psi_{yy} - \psi_{xx}) \cos 2\theta). \end{aligned} \quad (1.50)$$

Now using equation (1.34) in the equation (1.50) and then substituting the resulting equation in the equation (1.3), we finally get

$$\begin{aligned} \frac{D\psi}{Dt} = & M_\psi (\epsilon_0^2 \eta^2 \Delta \psi - (1-c)H_A(\psi, T) - cH_B(\psi, T)) \\ & - \frac{M_\psi \epsilon_0^2 (\eta \eta'' + (\eta')^2)}{2} \{2\psi_{xy} \sin 2\theta - \Delta \psi - (\psi_{yy} - \psi_{xx}) \cos 2\theta\} \\ & + M_\psi \epsilon_0^2 \eta \eta' \{ \sin 2\theta (\psi_{yy} - \psi_{xx}) + 2\psi_{xy} \cos 2\theta \}, \end{aligned} \quad (1.51)$$

which is the final form of the evolution equation for the phase-field $\psi(\mathbf{x}, t)$ in two dimensions, where we assume M_ψ be a positive constant.

In the next section, we shall present the derivation of the concentration and energy density equations which are coupled to the phase-field equation.

1.4 Energy and Concentration Equations

As described earlier the evolution equations for the concentration $c(\mathbf{x}, t)$ and the energy density $e(\mathbf{x}, t)$ are given by following normal conservation laws

$$\frac{Dc}{Dt} = -\text{div}(\mathbf{J}_c), \quad (1.52)$$

$$\frac{De}{Dt} = -\text{div}(\mathbf{J}_e), \quad (1.53)$$

where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material time derivative, \mathbf{J}_c and \mathbf{J}_e are the diffusional and heat flux respectively.

The fluxes \mathbf{J}_c and \mathbf{J}_e can be expressed by the irreversible linear laws as [24]

$$\mathbf{J}_c = M_c \nabla \frac{\delta S(\psi, c, e)}{\delta c}, \quad (1.54)$$

$$\mathbf{J}_e = M_e \nabla \frac{\delta S(\psi, c, e)}{\delta e}, \quad (1.55)$$

where the parameter M_c and M_e are assumed to be positive and are related to the A-B inter diffusion coefficient and the heat conductivity, respectively and $S(\psi, c, e)$ is the entropy functional which is defined by equation (1.6).

As observed by Warren and Boettinger [24] that the effects of the terms ∇T in the concentration equation (1.52) and ∇c in the energy equation (1.53) to be the small corrections. Therefore in the derivation of the concentration equation, we shall assume that the temperature $T(\mathbf{x}, t)$ is constant and in the derivation of equation of energy, the concentration $c(\mathbf{x}, t)$ will be assumed fixed.

To advance further we need to take the variational derivatives of S with respect to concentration $c(\mathbf{x}, t)$ and the energy density $e(\mathbf{x}, t)$. The variational derivatives of $S(\psi, c, e)$ (in the sense of distribution) with respect to c and e can easily be given using equation (1.6) as

$$\frac{\delta S(\psi, c, e)}{\delta c} = \frac{\partial s(\psi, c, e)}{\partial c}, \quad (1.56)$$

$$\frac{\delta S(\psi, c, e)}{\delta e} = \frac{\partial s(\psi, c, e)}{\partial e}. \quad (1.57)$$

According to the previously given relation (1.14) and using the basic thermodynamics, the two variational derivatives appearing in equations (1.56) and (1.57) can be expressed as

$$\frac{\partial s(\psi, c, e)}{\partial c} = -\frac{1}{T(\mathbf{x}, t)} \frac{\partial f(\psi, c, T)}{\partial c}, \quad (1.58)$$

$$\frac{\partial s(\psi, c, e)}{\partial e} = \frac{1}{T(\mathbf{x}, t)}. \quad (1.59)$$

By employing equations (1.16), (1.58) and (1.56), the equation (1.54) takes the form

$$\mathbf{J}_c = M_c \nabla \left(-\frac{\mu_B(\psi, c, T) - \mu_A(\psi, c, T)}{T(\mathbf{x}, t)} \right).$$

Since temperature $T(\mathbf{x}, t)$ is assumed to be constant in the derivation of the concentration equation therefore the above equation can be written as

$$\mathbf{J}_c = -\frac{M_c}{T} \nabla (\mu_B(\psi, c, T) - \mu_A(\psi, c, T)).$$

With the help of equations (1.17) and (1.18), the above equation takes the form

$$\begin{aligned} \mathbf{J}_c = & -\frac{M_c}{T} \nabla \left(f_B(\psi, T) + \frac{RT}{V_m} \ln(c) \right) \\ & + \frac{M_c}{T} \nabla \left(f_A(\psi, T) + \frac{RT}{V_m} \ln(1-c) \right). \end{aligned}$$

Carrying out the differentiation in the above equation, we have

$$\begin{aligned}\mathbf{J}_c = & -\frac{M_c}{T} \left(\nabla f_B(\psi, T) + \frac{RT}{V_m} \frac{1}{c} \nabla c \right) \\ & + \frac{M_c}{T} \left(\nabla f_A(\psi, T) + \frac{RT}{V_m} \frac{1}{1-c} (-\nabla c) \right).\end{aligned}$$

Making use of equation (1.33), we obtain

$$\begin{aligned}\mathbf{J}_c = & \frac{M_c}{T} \left(-W_B T g'(\psi) \nabla \psi - p'(\psi) L_B \left(1 - \frac{T}{T_m^B} \right) \nabla \psi \right) \\ & + \frac{M_c}{T} \left(W_A T g'(\psi) \nabla \psi + p'(\psi) L_A \left(1 - \frac{T}{T_m^A} \right) \nabla \psi \right) \\ & + \frac{M_c}{T} \left(-\frac{RT}{V_m} \frac{1}{c} \nabla c - \frac{RT}{V_m} \frac{1}{1-c} (\nabla c) \right),\end{aligned}$$

or

$$\begin{aligned}\mathbf{J}_c = & M_c \left(-W_B g'(\psi) - p'(\psi) L_B \left(\frac{1}{T} - \frac{1}{T_m^B} \right) \right) \nabla \psi \\ & + M_c \left(W_A g'(\psi) + p'(\psi) L_A \left(\frac{1}{T} - \frac{1}{T_m^A} \right) \right) \nabla \psi \\ & + M_c \left(-\frac{R}{V_m} \frac{1}{c} \nabla c - \frac{R}{V_m} \frac{1}{1-c} (\nabla c) \right).\end{aligned}$$

Using equations (1.35) and (1.36) in the above equation, we have

$$\begin{aligned}\mathbf{J}_c = & -M_c H_B(\psi, T) \nabla \psi + M_c H_A(\psi, T) \nabla \psi \\ & - M_c \frac{R}{V_m} \left(\frac{1}{c} + \frac{1}{1-c} \right) \nabla c.\end{aligned}$$

Further simplification of the above equation yields

$$\mathbf{J}_c = M_c (H_A(\psi, T) - H_B(\psi, T)) \nabla \psi - \frac{M_c R}{V_m c(1-c)} \nabla c. \quad (1.60)$$

Also as the comparison of equation (1.60) with the Fick's first law in a single-phase system (where $\nabla \psi = 0$) establishes the relation given below [24]

$$M_c = D(\psi) \frac{V_m c(1-c)}{R}, \quad (1.61)$$

where $D(\psi) = D_S + p(\psi) (D_L - D_S)$ is the A-B inter diffusion coefficient.

Substituting equation (1.61) into the equation (1.60), we obtain

$$\mathbf{J}_c = D(\psi) \frac{V_m c(1-c)}{R} (H_A(\psi, T) - H_B(\psi, T)) \nabla \psi - D(\psi) \nabla c. \quad (1.62)$$

Finally substituting equation (1.62) into the equation (1.52), we have

$$\frac{Dc}{Dt} = \text{div} \left(D(\psi) \left(\nabla c + \frac{c(1-c)V_m}{R} (H_B(\psi, T) - H_A(\psi, T)) \nabla \psi \right) \right). \quad (1.63)$$

Equation (1.63) represents the final form of the evolution equation for the mole fraction (concentration) of the solute.

If we assume that the interface thickness $\delta_A = \delta_B = \delta$, the above equation takes the form

$$\begin{aligned} \frac{Dc}{Dt} = \text{div} (D(\psi) \nabla c) + \text{div} \left(\alpha_0 D(\psi) c(1-c) \left\{ \frac{1}{\delta} \lambda'_1(c) g'(\psi) \right. \right. \\ \left. \left. + \lambda'_2(c) p'(\psi) \right\} \nabla \psi \right), \end{aligned} \quad (1.64)$$

where

$$\alpha_0 = \frac{3\sqrt{2}V_m}{RT_m} (\sigma_A + \sigma_B), \quad (1.65)$$

with $\lambda'_1(c) = \partial \lambda(c) / \partial c$, $\lambda'_2(c) = \partial \lambda(c) / \partial c$, where $\lambda_1(c)$ and $\lambda_2(c)$ are defined in equation (1.39).

Now we shall derive the evolution equation for the energy. For this, first, the internal energy density of a binary alloy can be expressed using a rule of mixture as (e.g., [24], [39])

$$e(\psi, c, T) = (1-c)e_A(\psi, T) + c e_B(\psi, T). \quad (1.66)$$

Making use of equation (1.59) into the equation (1.55) we get

$$\mathbf{J}_e = M_e \nabla \left(\frac{1}{T(\mathbf{x}, t)} \right),$$

or

$$\mathbf{J}_e = M_e \left(-\frac{1}{T^2} \nabla T(\mathbf{x}, t) \right). \quad (1.67)$$

Now substituting equations (1.66) and (1.67) in the equation (1.53), we have

$$\frac{D}{Dt} ((1-c)e_A(\psi, T) + c e_B(\psi, T)) = -\nabla \cdot \left(M_e \left(-\frac{1}{T^2} \nabla T(\mathbf{x}, t) \right) \right).$$

Since concentration $c(\mathbf{x}, t)$ is considered constant in the derivation of the energy equation, therefore the above equation take the form

$$(1-c) \frac{De_A(\psi, T)}{Dt} + c \frac{De_B(\psi, T)}{Dt} = -\nabla \cdot \left(M_e \left(-\frac{1}{T^2} \nabla T(\mathbf{x}, t) \right) \right).$$

By using equation (1.25) and setting $M_e = KT^2$, where K the thermal conductivity, the above equation becomes

$$(1 - c) \left(C_A \frac{DT}{Dt} + L_A \frac{Dp(\psi)}{Dt} \right) + c \left(C_B \frac{DT}{Dt} + L_B \frac{Dp(\psi)}{Dt} \right) = \nabla \cdot (K \nabla T).$$

Applying chain rule and re-arranging the above equation, we have

$$((1 - c)C_A + cC_B) \frac{DT}{Dt} + ((1 - c)L_A + cL_B) p'(\psi) \frac{D\psi}{Dt} = \nabla \cdot (K \nabla T).$$

Further the above equation can be written as ($p'(\psi) = 30g(\psi)$)

$$C \frac{DT(\mathbf{x}, t)}{Dt} + 30L g(\psi) \frac{D\psi(\mathbf{x}, t)}{Dt} = \nabla \cdot (K \nabla T(\mathbf{x}, t)), \quad (1.68)$$

where

$$\begin{aligned} C &= (1 - c)C_A + cC_B, \\ L &= (1 - c)L_A + cL_B, \\ K &= (1 - c)K_A + cK_B, \end{aligned}$$

with K_A and K_B , the thermal conductivities of substances A and B respectively.

Equation (1.68) represents the final form of the evolution equation for the temperature field $T(\mathbf{x}, t)$.

Next section is devoted to the derivation of the equations of melt flow which is coupled to the equations of phase-field, concentration and energy equations in the presence of a magnetic field.

1.5 Evolution Equations for the Melt Flow

As elucidated earlier that the evolution equations for the melt flow will be derived from the laws of conservation of momentum and mass. The domain Ω is initially occupied by the binary alloy of the substances A and B which is incompressible and electrically conducting fluid subject to applied magnetic field \mathbf{B}_m on the entire domain. The motion of the fluid is initially driven by the buoyancy force. Since the fluid is electrically conducting and also there is a applied magnetic field, therefore when the fluid start moving there would be electric current and in addition to the applied magnetic field, there will be induced magnetic field produced by the electric currents in the liquid metal. This will give rise to the Lorentz force which acts on the fluid so that an extra body force term \mathbf{F} will appear in the Navier-stokes equations. The Lorentz force in such a flow is given by as

$$\mathbf{F} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}_m, \quad (1.69)$$

where ρ_e is the electric charge density, \mathbf{E} the electric field intensity, \mathbf{J} is the current density and $\mathbf{B}_m = \mathbf{B} + \mathbf{b}$ is the sum of applied magnetic field \mathbf{B} and induced magnetic field \mathbf{b} .

We assume that the walls of the domain are electric insulators and the magnetic Reynolds number is sufficiently small that the induced magnetic field \mathbf{b} is negligible as compared to the imposed magnetic field \mathbf{B} (see e.g., [54]).

The current density \mathbf{J} appeared in equation (1.69) can be defined by the Ohm's law for the moving medium as

$$\mathbf{J} = \rho_e \mathbf{u} + \sigma_e (\mathbf{E} + \mathbf{u} \times \mathbf{B}), \quad (1.70)$$

where σ_e is electrical conductivity and \mathbf{u} is the velocity of the fluid.

Since electric field is a conservative field, therefore we can express it as a gradient of a scalar function ϕ as

$$\mathbf{E} = -\nabla \phi, \quad (1.71)$$

where ∇ is the gradient operator, ϕ is the potential function and negative sign shows that the electric field intensity always decreases from higher to lower potential.

Thus the equations (1.69) and (1.70) together with equation (1.71) takes the form

$$\mathbf{F} = -\rho_e \nabla \phi + \sigma_e (-\nabla \phi + \mathbf{u} \times \mathbf{B}) \times \mathbf{B}, \quad (1.72)$$

$$\mathbf{J} = \rho_e \mathbf{u} + \sigma_e (-\nabla \phi + \mathbf{u} \times \mathbf{B}). \quad (1.73)$$

In addition to the Ohm's law, the current density \mathbf{J} is governed by the conservation of electric current

$$\text{div}(\mathbf{J}) = 0. \quad (1.74)$$

This equation can be used to calculate the potential function ϕ appearing in the Lorentz force \mathbf{F} . Taking the divergence on both sides of the equation (1.73), we have

$$\text{div}(\mathbf{J}) = \text{div}(\rho_e \mathbf{u}) + \text{div}(\sigma_e (-\nabla \phi + \mathbf{u} \times \mathbf{B})).$$

Since electric charged density ρ_e and electrical conductivity σ_e are assumed to be constant here and using equation (1.74), we have

$$\rho_e \text{div}(\mathbf{u}) + \sigma_e \text{div}(-\nabla \phi + \mathbf{u} \times \mathbf{B}) = 0,$$

Using incompressibility condition *i.e.*, $\text{div}(\mathbf{u}) = 0$, we arrive at

$$\Delta \phi = \text{div}(\mathbf{u} \times \mathbf{B}), \quad (1.75)$$

where Δ is the Laplace operator.

From the above equation, we can calculate the potential function ϕ under the influence of magnetic field applied in any direction. Therefore with the help of this potential along with the magnetic field we can calculate the Lorentz force \mathbf{F} defined in equation (1.72).

Also note that to derive equations for the melt flow, we shall assume the Boussinesq approximations, as is often done in the heat and/or solute transfer problems. This will lead us to neglect the density variations with respect to temperature and/or concentration everywhere except in the gravitational force term in the momentum equation, and also neglecting the temperature variations of the other material properties.

Also as we know that the phase-field variable $\psi(\mathbf{x}, t)$ is 0 in the solid phase and 1 in the liquid phase and there is no motion in the solid phase, therefore equations of the melt flow should give us the *zero* velocity in the solid region of the domain. To include this fact in the equations of melt flow, we have multiplied the boussinesq approximation term and Lorentz force term by functions $a_1(\psi)$ and $a_2(\psi)$. These functions are chosen in way that they are 0 at $\psi(\mathbf{x}, t) = 0$ and 1 at $\psi(\mathbf{x}, t) = 1$, so that the Boussinesq approximation term and Lorentz force term become zero in the solid region and the equations of the melt flow together with the zero initial and boundary conditions give the *zero* velocity in the solid region of the domain. Also to include the effects on the velocity with respect to the phase change at the solid/liquid interface, we have added an additional term $\mathbf{f}(\psi)$ in the melt flow equations which will also be chosen so that it is *zero* at $\psi(\mathbf{x}, t) = 0$. The melt flow equations can be given using incompressible Navier-stokes equations as

$$\rho_0 \frac{D\mathbf{u}}{Dt} = \text{div}(\vec{\sigma}) + a_1(\psi) (-\beta_T T(\mathbf{x}, t) - \beta_c c(\mathbf{x}, t)) \mathbf{G} + a_2(\psi) \sigma_e (-\nabla \phi + \mathbf{u} \times \mathbf{B}) \times \mathbf{B} + \alpha \mathbf{f}(\psi), \quad (1.76)$$

$$\text{div}(\mathbf{u}) = 0. \quad (1.77)$$

Which is a magnetohydrodynamic type system with $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, ρ_0 is the mean density of the fluid, β_T and β_c are the thermal and solutal expansion coefficients, $\mathbf{G} = (0, 0, -g)$ is the gravity vector, $T(\mathbf{x}, t)$ is the temperature, $c(\mathbf{x}, t)$ is the concentration (mole fraction of the substance B in A) and $\vec{\sigma}$ is the stress tensor which is defined as

$$\vec{\sigma} = -p\mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^{tran}) \quad (1.78)$$

where p is the pressure, \mathbf{I} is the unit tensor, μ is the dynamic viscosity, and *tran* represents the usual transpose of a matrix.

Remark: The functions $a_1(\psi)$ and $a_2(\psi)$ in the equation (1.76) can be chosen, for example, as

$$a_1(\psi) = \psi, \quad a_2(\psi) = \frac{\psi(1 + \psi)}{2}. \quad (1.79)$$

In the next section we shall give all equations that governs the solidification process of the binary alloy.

1.6 Mathematical Models

In this section, we shall summarize the entire set of governing equations that model the solidification process of a binary alloy in a non-isothermal environment in the presence of motion in the liquid phase with the magnetic field effect. The equations that model this phenomenon are the phase-field equation (1.37), concentration equation (1.63), energy equation (1.68) and the magnetohydrodynamic system (1.75-1.77) which are given below

$$\begin{aligned} \frac{D\psi}{Dt} = M_\psi \Big(\operatorname{div} (\epsilon_\theta^2 \nabla \psi) - (1 - c)H_A(\psi, T) - cH_B(\psi, T) \\ - A\left(\epsilon_\theta, \epsilon'_\theta, \frac{\partial \theta}{\partial \psi}, \nabla \psi\right) \Big) \quad \text{on } \mathcal{Q}, \end{aligned} \quad (1.80)$$

$$\frac{Dc}{Dt} = \operatorname{div} \left(D(\psi) \left(\nabla c + \frac{c(1 - c)V_m}{R} (H_B(\psi, T) - H_A(\psi, T)) \nabla \psi \right) \right) \quad \text{on } \mathcal{Q}, \quad (1.81)$$

$$C \frac{DT}{Dt} + 30L g(\psi) \frac{\partial \psi}{\partial t} = \nabla \cdot (K \nabla T) \quad \text{on } \mathcal{Q}, \quad (1.82)$$

$$\begin{aligned} \rho_0 \frac{D\mathbf{u}}{Dt} = \operatorname{div}(\vec{\sigma}) + a_1(\psi) (-\beta_T T(\mathbf{x}, t) - \beta_c c(\mathbf{x}, t)) \mathbf{G} \\ + a_2(\psi) \sigma_e (-\nabla \phi + \mathbf{u} \times \mathbf{B}) \times \mathbf{B} + \alpha \mathbf{f}(\psi) \quad \text{on } \mathcal{Q}, \end{aligned} \quad (1.83)$$

$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{on } \mathcal{Q}, \quad (1.84)$$

$$\vec{\sigma} = -p\mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^{tran}), \quad (1.85)$$

$$\Delta \phi = \operatorname{div}(\mathbf{u} \times \mathbf{B}) \quad \text{on } \mathcal{Q}, \quad (1.86)$$

where all the variables and parameters are defined in the respective sections of the derivation of the equations.

In the next subsection, the developed model will be restricted to the isothermal case.

1.6.1 Isothermal-Anisotropic Case

An isothermal process is a process in which the temperature of the system remains unchanged *i.e.* there is no change in the temperature in the whole process. Therefore in the isothermal case of our model the temperature $T(\mathbf{x}, t)$ is assumed to be constant in the whole model. Therefore we shall not consider the evolution equation for the temperature in this case. The model problem in this case reduces to

$$\begin{aligned} \frac{D\psi}{Dt} = M_\psi \Big(\operatorname{div} (\epsilon_\theta^2 \nabla \psi) - (1-c) \tilde{H}_A(\psi) - c \tilde{H}_B(\psi) \\ - A \left(\epsilon_\theta, \epsilon'_\theta, \frac{\partial \theta}{\partial \psi}, \nabla \psi \right) \Big) \quad \text{on } \mathcal{Q}, \end{aligned} \quad (1.87)$$

$$\frac{Dc}{Dt} = \operatorname{div} \left(D(\psi) \left(\nabla c + \frac{c(1-c)V_m}{R} \left(\tilde{H}_B(\psi) - \tilde{H}_A(\psi) \right) \nabla \psi \right) \right) \quad \text{on } \mathcal{Q}, \quad (1.88)$$

$$\begin{aligned} \rho_0 \frac{D\mathbf{u}}{Dt} = \operatorname{div}(\vec{\sigma}) - a_1(\psi) \beta_c c(\mathbf{x}, t) \mathbf{G} \\ + a_2(\psi) \sigma_e (-\nabla \phi + \mathbf{u} \times \mathbf{B}) \times \mathbf{B} + \alpha \mathbf{f}(\psi) \quad \text{on } \mathcal{Q}, \end{aligned} \quad (1.89)$$

$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{on } \mathcal{Q}, \quad (1.90)$$

$$\vec{\sigma} = -p\mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^{tran}), \quad (1.91)$$

$$\Delta \phi = \operatorname{div}(\mathbf{u} \times \mathbf{B}) \quad \text{on } \mathcal{Q}, \quad (1.92)$$

where $\tilde{H}_I(\psi) = H_I(\psi, T)$, $I = A, B$.

We suppose that the physical system where solidification process takes place is a closed system, that is, there is no phase and concentration exchange across the boundary and that the velocity in the liquid phase along the boundary is negligible. Therefore we have enclosed the geometry of the problem by posing Neumann boundary conditions for the phase-field and concentration equations and no-slip condition for the flow equations along with the initial conditions given below

$$(\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0) \quad \text{in } \Omega, \quad (1.93a)$$

$$\mathbf{u} = 0, \quad \frac{\partial \psi}{\partial \mathbf{n}} = \nabla \psi \cdot \mathbf{n} = 0, \quad \frac{\partial c}{\partial \mathbf{n}} = \nabla c \cdot \mathbf{n} = 0 \quad \text{on } \Sigma = (0, T_f) \times \partial \Omega, \quad (1.93b)$$

where T_f is the final time, \mathbf{n} is the unit outward normal and Ω is a sufficiently regular and open domain in dimension \mathbb{R}^n with $n \leq 3$.

Next we shall reduce our model problem defined above to the two dimensional case in an isothermal environment.

1.6.2 Two Dimensional Isothermal-Anisotropic Case

In a two dimensional case, we work in the XZ-plane and suppose that the applied magnetic field \mathbf{B} is parallel to the plane. Thus the equation (1.75) reduces to

$$\Delta\phi = 0, \quad (1.94)$$

which is valid in the melt as well as in the neighboring solid media. This condition alongwith the electrically insulating conditions on the boundary implies that the unique solution is $\nabla\phi = 0$, and therefore the electric field vanishes everywhere. In such situations, the equation related to u_2 component of velocity and the electric potential equation (1.75) in the magnetohydrodynamic system will be decoupled from other equations. Therefore equation (1.73) reduces to

$$\mathbf{J} = \sigma_e(\mathbf{u} \times \mathbf{B}), \quad (1.95)$$

and the Lorentz force defined in the equation (1.72) takes the form

$$\mathbf{F} = \sigma_e(\mathbf{u} \times \mathbf{B}) \times \mathbf{B}. \quad (1.96)$$

Using equations (1.95), (1.96) and the equation (1.51) the model problem in two dimensions for the isothermal anisotropic case reduces to the following form

$$\begin{aligned} \rho_0 \frac{D\mathbf{u}}{Dt} = & -\nabla p + \mu\Delta\mathbf{u} + a_1(\psi)\beta_c c(\mathbf{x}, t)\mathbf{G} \\ & + a_2(\psi)\sigma_e(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} + \alpha\mathbf{f}(\psi) \quad \text{on } \mathcal{Q}, \end{aligned} \quad (1.97)$$

$$\text{div}(\mathbf{u}) = 0 \quad \text{on } \mathcal{Q}, \quad (1.98)$$

$$\begin{aligned} \frac{D\psi}{Dt} = & M_\psi \left(\epsilon_0^2 \eta^2 \Delta\psi - (1-c)\tilde{H}_A(\psi) - c\tilde{H}_B(\psi) \right) \\ & - \frac{M_\psi \epsilon_0^2 (\eta\eta'' + (\eta')^2)}{2} \{ 2\psi_{xy} \sin 2\theta - \Delta\psi - (\psi_{yy} - \psi_{xx}) \cos 2\theta \} \\ & + M_\psi \epsilon_0^2 \eta\eta' \{ \sin 2\theta (\psi_{yy} - \psi_{xx}) + 2\psi_{xy} \cos 2\theta \} \quad \text{on } \mathcal{Q}, \end{aligned} \quad (1.99)$$

$$\frac{Dc}{Dt} = \text{div} \left(D(\psi) \left(\nabla c + \frac{c(1-c)V_m}{R} \left(\tilde{H}_B(\psi) - \tilde{H}_A(\psi) \right) \nabla\psi \right) \right) \quad \text{on } \mathcal{Q}, \quad (1.100)$$

where, for simplicity, we denote $\mathbf{x} = (x, y)$, $\mathbf{u} = (u, v)$ and $\mathbf{B} = (B_1, B_2)$ and to simplify further the notation, we write the above mentioned problem in the following

form

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{A}_1(\psi, c) + b(\psi)(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \quad \text{on } \mathcal{Q}, \quad (1.101a)$$

$$\text{div}(\mathbf{u}) = 0, \quad \text{on } \mathcal{Q}, \quad (1.101b)$$

$$\frac{D\psi}{Dt} = \gamma(\eta) \Delta \psi - A_2(\psi, c) - A_4(\eta, \eta', \eta'', \nabla \psi, \nabla(\nabla \psi)) \quad \text{on } \mathcal{Q}, \quad (1.101c)$$

$$\frac{Dc}{Dt} = \text{div}(D(\psi) \nabla c) + \text{div}(A_3(\psi, c) \nabla \psi) \quad \text{on } \mathcal{Q}, \quad (1.101d)$$

with the initial and boundary conditions

$$(\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0) \quad \text{in } \Omega, \quad (1.102a)$$

$$\mathbf{u} = 0, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \frac{\partial c}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma = (0, T_f) \times \partial \Omega, \quad (1.102b)$$

where $\gamma(\eta) = M_\psi \epsilon_0^2 \eta^2$, $b(\psi) = \sigma_e a_2(\psi)$ and

$$\begin{aligned} \mathbf{A}_1(\psi, c) &= a_1(\psi) \beta_c c(\mathbf{x}, t) \mathbf{G} + \alpha \mathbf{f}(\psi), \\ A_2(\psi, c) &= M_\psi \left((1-c) \tilde{H}_A(\psi) - c \tilde{H}_B(\psi) \right), \\ A_3(\psi, c) &= \frac{c(1-c)V_m}{R} \left(\tilde{H}_B(\psi) - \tilde{H}_A(\psi) \right), \\ A_4(\eta, \eta', \eta'', \nabla \psi, \nabla(\nabla \psi)) &= \frac{M_\psi \epsilon_0^2 (\eta \eta'' + (\eta')^2)}{2} \{ 2\psi_{xy} \sin 2\theta - \Delta \psi - (\psi_{yy} - \psi_{xx}) \cos 2\theta \} \\ &\quad - M_\psi \epsilon_0^2 \eta \eta' \{ \sin 2\theta (\psi_{yy} - \psi_{xx}) + 2\psi_{xy} \cos 2\theta \}. \end{aligned}$$

Further in the next subsection, we shall give a isotropic case of the above defined model.

1.6.3 Two dimensional Isothermal and Isotropic Case

In the model derived above the interfacial energy parameter $\epsilon = \epsilon_0 \eta$ is assumed to be anisotropic. In this case the process of solidification is dependent on the direction of the solid/liquid interface. In an isotropic case this parameter does not depend on the direction of the solid/liquid interface and is assumed to be constant, *i.e.* η is constant, therefore the operator A_4 will become zero and the process of solidification is homogeneous in all directions in this case. In the isothermal and isotropic case our model problem will be reduced to take the following form

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{A}_1(\psi, c) + b(\psi)(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \quad \text{on } \mathcal{Q}, \quad (1.103a)$$

$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{on } \mathcal{Q}, \quad (1.103b)$$

$$\frac{D\psi}{Dt} = \epsilon_1 \Delta \psi - A_2(\psi, c) \quad \text{on } \mathcal{Q}, \quad (1.103c)$$

$$\frac{Dc}{Dt} = \operatorname{div}(D(\psi)\nabla c) + \operatorname{div}(A_3(\psi, c)\nabla \psi) \quad \text{on } \mathcal{Q}, \quad (1.103d)$$

where $\epsilon_1 = \gamma(\eta = 1)$.

The initial and boundary conditions are defined same as in (1.93)

$$(\mathbf{u}, \psi, c)(t = 0) = (\mathbf{u}_0, \psi_0, c_0) \quad \text{in } \Omega, \quad (1.104a)$$

$$\mathbf{u} = 0, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \frac{\partial c}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma = (0, T_f) \times \partial\Omega. \quad (1.104b)$$

In the next chapter, we shall present the existence, uniqueness and regularity of the solutions of the problem (1.103) under some assumptions on the non-linear operators.

Chapter 2

Existence, Regularity and Stability Results

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2.1 Introduction

In this chapter, we shall present the existence, regularity and stability results of the two dimensional isothermal-isotropic case of our model problem defined by (1.103) using some a priori results, elliptic estimates and by posing some conditions on the non-linear operators. In section 2.2, we shall define some fundamental spaces to be used in the rest of the chapter. In section 2.3, we shall present the assumptions made on the non-linear operators and the weak formulation of the model problem will be given in the section 2.4. In section 2.5 and 2.6, we shall provide the proofs of theorem which shows the existence, regularity and uniqueness of the problem.

2.2 Definitions and Notations

Let Ω be a fixed bounded and open domain in \mathbb{R}^2 and $\Gamma = \partial\Omega$ denotes its boundary which is supposed to be sufficiently regular.

For $p \in [1, +\infty]$, we denote by $L^p(\Omega)$ the space of p-integrable functions with the norm

$$\|v\|_{L^p(\Omega)} = \left(\int_{\Omega} |v(\mathbf{x})|_2^p d\mathbf{x} \right)^{1/p}, \quad \text{if } p < \infty$$

$$\|v\|_{L^\infty(\Omega)} = \sup_{\mathbf{x} \in \Omega} \text{ess } |v(\mathbf{x})|_2, \quad \text{if } p = \infty$$

In particular for $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_{\Omega} u(\mathbf{x}) \cdot v(\mathbf{x}) d\mathbf{x}, \quad \forall u, v \in L^2(\Omega)$$

and we shall denote the norm of $L^2(\Omega)$ as

$$|v| = \left(\int_{\Omega} |v(\mathbf{x})|_2^2 d\mathbf{x} \right)^{1/2}$$

where $|\cdot|_2$ is the usual Euclidean norm.

For an integer $m > 0$ and $1 \leq p \leq \infty$, the Sobolev space of order (m, p) , denoted by $W^{m,p}(\Omega)$, is defined as the space of functions in the space $L^p(\Omega)$ whose derivatives upto order $\leq m$ are also in $L^p(\Omega)$, that is

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \forall \alpha \in \mathbb{N}^n \text{ such that } [\alpha] \leq m\}$$

where $\alpha = (\alpha_1, \dots, \alpha_j, \dots, \alpha_n)$ and $[\alpha] = \sum_i \alpha_i$, together with the norm

$$\|v\|_{W^{m,p}(\Omega)} = \left(\sum_{[\alpha] \leq m} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p}, \quad \text{if } 1 \leq p < \infty$$

and

$$\|v\|_{W^{m,\infty}(\Omega)} = \max_{[\alpha] \leq m} \|D^\alpha v\|_{L^\infty(\Omega)}$$

In particular, if $p = 2$, the Sobolev space $W^{m,2}(\Omega)$, denoted by $H^m(\Omega)$, is a Hilbert space for the following scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{[\alpha] \leq m} (D^\alpha u, D^\alpha v)$$

where (\cdot, \cdot) is the scalar product of $L^2(\Omega)$. We shall denote the norm of $H^m(\Omega)$ as

$$\|v\|_{H^m(\Omega)} = \left(\sum_{[\alpha] \leq m} \|D^\alpha v\|_{L^2(\Omega)}^2 \right)^{1/2}$$

Further if $m = 1$, the space $H^1(\Omega)$ can be defined as

$$H^1(\Omega) = \{v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega)\}$$

with the scalar product defined as

$$(u, v)_{H^1(\Omega)} = \int_{\Omega} u(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \int_{\Omega} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})d\mathbf{x}$$

and the norm on $H^1(\Omega)$ is denoted by

$$\|v\| = (|v(\mathbf{x})|^2 + |\nabla v(\mathbf{x})|^2)^{1/2}$$

Let $\mathcal{D}(\Omega)$ (or $\mathcal{D}(\overline{\Omega})$) be the space of \mathcal{C}^∞ functions with compact support contained in Ω (or $\overline{\Omega}$). The closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$ (or $H_0^m(\Omega)$ if $p = 2$).

In particular, the space $H_0^1(\Omega)$ is defined as

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma\}$$

with the scalar product defined as

$$(u, v)_{H_0^1(\Omega)} = \int_{\Omega} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})d\mathbf{x}$$

and the norm on $H_0^1(\Omega)$ is defined by $\|v(\mathbf{x})\|_{H_0^1(\Omega)} = |\nabla v(\mathbf{x})|$.

The dual space of the space $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ with the norm defined by

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|\langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}|}{\|v\|}$$

Let U be a Banach space and $1 \leq p \leq +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions v from (a, b) into U which is a Banach space with the norm

$$\|v\|_{L^p(a, b; U)} = \left(\int_a^b \|v\|_U^p dt \right)^{1/p}, \quad \text{if } p < \infty$$

and

$$\|v\|_{L^\infty(a, b; U)} = \sup_{t \in (a, b)} \text{ess } \|v\|_U, \quad \text{if } p = +\infty$$

Now we shall define some basic spaces which will be used frequently in the study of our problem.

$$\begin{aligned} H &= \left\{ \mathbf{v} \in (L^2(\Omega))^2 \mid \text{div}(\mathbf{v}) = 0 \right\} \\ V &= \left\{ \mathbf{v} \in (H^1(\Omega))^2 \mid \text{div}(\mathbf{v}) = 0, \mathbf{v} = 0 \text{ on } \Gamma \right\} \\ H_0^2 &= \left\{ v \in H^2(\Omega) \mid \frac{\partial v}{\partial \mathbf{n}} = 0 \right\} \\ \mathbf{H} &= H \times L^2(\Omega) \times L^2(\Omega), \quad \mathbf{V} = V \times H^1(\Omega) \times H^1(\Omega) \end{aligned}$$

We then define the Leray projection \mathbb{P} to be the orthogonal projection of $(L^2(\Omega))^2$ onto H . Using divergence theorem it can easily be proven that any gradient is orthogonal to H , therefore if we apply \mathbb{P} to the equation (1.103a), the pressure term will be eliminated and we shall left with a evolutionary parabolic equation.

Also note that the vector triple product of three vectors $\mathbf{f} = (f_1, f_2)$, $\mathbf{g} = (g_1, g_2)$, $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$ is defined by

$$(\mathbf{f} \times \mathbf{g}) \times \mathbf{h} = \begin{pmatrix} f_2 g_1 h_2 - f_1 g_2 h_2 \\ f_1 g_2 h_1 - f_2 g_1 h_1 \end{pmatrix}. \quad (2.1)$$

2.3 Assumptions

We state the following assumptions for the operators \mathbf{A}_1 , D , A_2 and A_3 in the problem (1.103) (see [6], [8]).

(H1) $\mathbf{A}_1(\mathbf{x}, t, \cdot)$ is a Caratheodory function from $\mathcal{Q} \times \mathbb{R}^2$ into \mathbb{R}^2 . For almost all $(\mathbf{x}, t) \in \mathcal{Q}$, $\mathbf{A}_1(\mathbf{x}, t, \cdot)$ is a Lipschitz positive and bounded function with

$$0 < a_0 \leq |\mathbf{A}_1(\mathbf{x}, t, \mathbf{r})|_2 \leq a_1, \quad \forall \mathbf{r} \in \mathbb{R}^2 \text{ and a.e. } (\mathbf{x}, t) \in \mathcal{Q}.$$

(H2) $A_2(\mathbf{x}, t, \cdot)$ and $A_3(\mathbf{x}, t, \cdot)$ are the Caratheodory functions from $\mathcal{Q} \times \mathbb{R}^2$ into \mathbb{R} . For almost all $(x, t) \in \mathcal{Q}$, $A_2(\mathbf{x}, t, \cdot)$ and $A_3(\mathbf{x}, t, \cdot)$ are Lipschitz positive and bounded functions with

$$0 < \tilde{a}_i \leq A_i(\mathbf{x}, t, \mathbf{r}) \leq a_i, \quad \forall i = 2, 3, \quad \forall \mathbf{r} \in \mathbb{R}^2 \text{ and a.e. } (\mathbf{x}, t) \in \mathcal{Q}.$$

(H3) $D(\mathbf{x}, t, \cdot)$ is a Caratheodory function from $\mathcal{Q} \times \mathbb{R}$ into \mathbb{R} . For almost all $(x, t) \in \mathcal{Q}$, $D(\mathbf{x}, t, \cdot)$ is Lipschitz positive and bounded function with

$$0 < D_0 \leq D(\mathbf{x}, t, r) \leq D_1, \quad \forall r \in \mathbb{R} \text{ and a.e. } (\mathbf{x}, t) \in \mathcal{Q}.$$

(H4) $b(\mathbf{x}, t, \cdot)$ is a Caratheodory function from $\mathcal{Q} \times \mathbb{R}$ into \mathbb{R} . For almost all $(x, t) \in \mathcal{Q}$, $b(\mathbf{x}, t, \cdot)$ is Lipschitz positive and bounded function with

$$0 < b_0 \leq b(\mathbf{x}, t, r) \leq b_1, \quad \forall r \in \mathbb{R} \text{ and a.e. } (\mathbf{x}, t) \in \mathcal{Q}.$$

(H5) $\mathbf{B} \in \{\mathbf{B} \in (L^2(\Omega))^2 \mid B_1 \leq |\mathbf{B}|_2 \leq B_2\} \subset (L^\infty(\mathcal{Q}))^2$.

For the sake of simplicity, we shall write $\mathbf{A}_1(\psi, c)$, $A_i(\psi, c)$, $D(\psi)$ and $b(\psi)$ in stead of $\mathbf{A}_1(\mathbf{x}, t, \psi, c)$, $A_i(\mathbf{x}, t, \psi, c)$, $D(\mathbf{x}, t, \psi)$ and $b(\mathbf{x}, t, \psi)$, $i = 2, 3$, respectively.

2.4 Weak Formulation

In this section we shall derive the weak formulation of the problem (1.103) together with the initial and boundary conditions (1.93). Applying Leray projection \mathbb{P} onto the equation (1.103a), we obtain

$$\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + B(\mathbf{u}, \mathbf{u}) \right) = \mu \mathbb{P} \Delta \mathbf{u} + \mathbb{P} \mathbf{A}_1(\psi, c) + \mathbb{P} (b(\psi) ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B})), \quad (2.2a)$$

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla) \psi = \epsilon_1 \Delta \psi - A_2(\psi, c), \quad \text{on } \mathcal{Q} \quad (2.2b)$$

$$\frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla) c = \text{div}(D(\psi) \nabla c) + \text{div}(A_3(\psi, c) \nabla \psi), \quad \text{on } \mathcal{Q} \quad (2.2c)$$

together with the initial and boundary conditions

$$(\mathbf{u}, \psi, c)(t = 0) = (\mathbf{u}_0, \psi_0, c_0), \quad \text{in } \Omega \quad (2.3a)$$

$$\mathbf{u} = 0, \quad \frac{\partial \psi}{\partial \mathbf{n}} = \nabla \psi \cdot \mathbf{n} = 0, \quad \frac{\partial c}{\partial \mathbf{n}} = \nabla c \cdot \mathbf{n} = 0, \quad \text{on } \Sigma = (0, T_f) \times \partial \Omega \quad (2.3b)$$

where $B(\mathbf{u}, \mathbf{u}) = \mathbb{P}(\mathbf{u} \cdot \nabla) \mathbf{u}$ and T_f is the final time. Now we define bilinear forms

$$a_u : V \times V \rightarrow \mathbb{R}$$

defined by

$$a_u(\mathbf{u}, \mathbf{v}) = \mu \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

and

$$a_\psi : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$$

defined by

$$a_\psi(\psi, \phi) = \epsilon_1 \int_{\Omega} \nabla \psi \cdot \nabla \phi \, d\mathbf{x}, \quad \forall \psi, \phi \in H^1(\Omega),$$

and a trilinear form as

$$b_u : V \times V \times V \rightarrow \mathbb{R}$$

defined by

$$b_u(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \rho_0 \sum_{i=1}^2 \sum_{j=1}^2 \int u_i (\partial_i v_j) w_j \, d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V.$$

Also note that $b_u(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \rho_0(B(\mathbf{u}, \mathbf{v}), \mathbf{w})$ and

$$b_\psi : V \times H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$$

defined by

$$b_\psi(\mathbf{u}; \psi, \phi) = \sum_{i=1}^2 \int u_i (\partial_i \psi) \phi \, d\mathbf{x}, \quad \forall \mathbf{u} \in V, \psi, \phi \in H^1(\Omega),$$

and in the same manner, we define

$$b_c(\mathbf{u}; c, z) = \sum_{i=1}^2 \int u_i (\partial_i c) z \, d\mathbf{x}, \quad \forall \mathbf{u} \in V, c, z \in H^1(\Omega).$$

Multiplying equation (2.2a) by $\mathbf{v} \in V$, equation (2.2b) by $\phi \in H^1(\Omega)$ and equation (2.2c) by $z \in H^1(\Omega)$, and then integrating the resulting equations over Ω , we obtain

$$\begin{aligned} \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + b_u(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= \mu(\Delta \mathbf{u}, \mathbf{v}) + (\mathbf{A}_1(\psi, c), \mathbf{v}) \\ &\quad + (b(\psi)((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}), \mathbf{v}), \end{aligned} \quad (2.4)$$

$$\left(\frac{\partial \psi}{\partial t}, \phi \right) + b_\psi(\mathbf{u}, \psi, \phi) = \epsilon_1(\Delta \psi, \phi) - (A_2(\psi, c), \phi), \quad (2.5)$$

$$\begin{aligned} \left(\frac{\partial c}{\partial t}, z \right) + b_c(\mathbf{u}, c, z) &= \int_{\Omega} \operatorname{div}(D(\psi) \nabla c) z \, d\mathbf{x} \\ &\quad + \int_{\Omega} \operatorname{div}(A_3(c, \psi) \nabla \psi) z \, d\mathbf{x}. \end{aligned} \quad (2.6)$$

Consider the integral

$$\int_{\Omega} \operatorname{div}(\mathbf{v} \cdot \nabla \mathbf{u}) \, d\mathbf{x} = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mathbf{v} \cdot \Delta \mathbf{u} \, d\mathbf{x},$$

by using the divergence theorem on the left-hand-side of the above equation, we have

$$\int_{\Gamma} (\mathbf{v} \cdot \nabla \mathbf{u}) \cdot \mathbf{n} \, d\Gamma = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mathbf{v} \cdot \Delta \mathbf{u} \, d\mathbf{x},$$

since $\mathbf{v} = \mathbf{0}$ on the boundary Γ , therefore we get

$$\int_{\Omega} \mathbf{v} \cdot \Delta \mathbf{u} \, d\mathbf{x} = - \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x}. \quad (2.7)$$

Similarly we can easily derive using $\nabla \psi \cdot \mathbf{n} = 0$ that

$$\int_{\Omega} \phi \Delta \psi \, d\mathbf{x} = - \int_{\Omega} \nabla \psi \cdot \nabla \phi \, d\mathbf{x}. \quad (2.8)$$

Consider the following integral

$$\int_{\Omega} \operatorname{div}(z D(\psi) \nabla c) \, d\mathbf{x} = \int_{\Omega} D(\psi) \nabla c \cdot \nabla z \, d\mathbf{x} + \int_{\Omega} z \operatorname{div}(D(\psi) \nabla c) \, d\mathbf{x},$$

applying divergence theorem on the left-hand-side of the above equation, we get

$$\int_{\Gamma} z D(\psi) \nabla c \cdot \mathbf{n} \, d\Gamma = \int_{\Omega} D(\psi) \nabla c \cdot \nabla z \, d\mathbf{x} + \int_{\Omega} z \operatorname{div}(D(\psi) \nabla c) \, d\mathbf{x},$$

since $\nabla c \cdot \mathbf{n} = 0$, then

$$\int_{\Omega} z \operatorname{div}(D(\psi) \nabla c) \, d\mathbf{x} = - \int_{\Omega} D(\psi) \nabla c \cdot \nabla z \, d\mathbf{x}. \quad (2.9)$$

Similarly, we have the following result (using $\nabla \psi \cdot \mathbf{n} = 0$),

$$\int_{\Omega} z \operatorname{div}(A_3(\psi, c) \nabla \psi) \, d\mathbf{x} = - \int_{\Omega} A_3(\psi, c) \nabla \psi \cdot \nabla z \, d\mathbf{x}. \quad (2.10)$$

Now making use of equations (2.7)-(2.10) in the equations (2.4)-(2.6), we arrive at

$$\left. \begin{aligned} & \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + a_u(\mathbf{u}, \mathbf{v}) + b_u(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{A}_1(\psi, c) \cdot \mathbf{v} \, d\mathbf{x} \\ & \quad + \int_{\Omega} b(\psi) ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{v} \, d\mathbf{x}, \\ & \left(\frac{\partial \psi}{\partial t}, \phi \right) + a_{\psi}(\psi, \phi) + b_{\psi}(\mathbf{u}, \psi, \phi) = - \int_{\Omega} A_2(\psi, c) \phi \, d\mathbf{x}, \\ & \left(\frac{\partial c}{\partial t}, z \right) + b_c(\mathbf{u}, c, z) + \int_{\Omega} D(\psi) \nabla c \cdot \nabla z \, d\mathbf{x} \\ & \quad + \int_{\Omega} A_3(\psi, c) \nabla \psi \cdot \nabla z \, d\mathbf{x} = 0, \quad \forall (\mathbf{v}, \phi, z) \in \mathbf{V} \\ & (\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0), \end{aligned} \right\} \quad (2.11)$$

which is the final form of the weak formulation of the problem (1.103).

Before proving the existence and uniqueness, we shall give some lemmas.

Lemma 1 (i) **Elliptic Estimate:** Let $k \in \mathbb{N}$ and $v \in H^2(\Omega)$ satisfy $\Delta v \in H^k(\Omega)$ and $\frac{\partial v}{\partial \mathbf{n}} = 0$ on the boundary Γ . Then $v \in H^{k+2}(\Omega)$ and we have the following estimate: there exists a constant $C > 0$ independent of v such that

$$\|v\|_{H^{k+2}(\Omega)} \leq C \left(\|\Delta v\|_{H^k(\Omega)} + \|v\|_{H^k(\Omega)} \right).$$

(ii) **Gagliardo-Nirenberg's inequality:** There exist a constant $C > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq C \|v\|_{H^q(\Omega)}^\theta \|v\|_{L^2(\Omega)}^{1-\theta} \quad \forall v \in H^q(\Omega),$$

where $0 \leq \theta < 1$ and $p = \frac{2n}{n-2\theta q}$, with the exception that if $q - n/2$ is a nonnegative integer then θ is restricted to zero.

(iii) For any set $\Omega \subset \mathbb{R}^2$, we have

$$\|v\|_{L^4(\Omega)} \leq C \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} \quad \forall v \in H_0^1(\Omega).$$

For the proof of this lemma, see, e.g., [23] and [55]. \square

Lemma 2 For any open set Ω , then the trilinear forms have the following properties

(i) For all $\mathbf{u} \in V$, $\mathbf{v} \in (H_0^1(\Omega))^2$ and $\psi, c \in H^1(\Omega)$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad b(\mathbf{u}, \psi, \psi) = 0, \quad b(\mathbf{u}, c, c) = 0.$$

(ii) For all $\mathbf{u} \in V$, $\mathbf{v}, \mathbf{w} \in (H_0^1(\Omega))^2$, $\psi, \phi \in H^1(\Omega)$ and $c, z \in H^1(\Omega)$.

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad b(\mathbf{u}, \psi, \phi) = -b(\mathbf{u}, \phi, \psi), \quad b(\mathbf{u}, c, z) = -b(\mathbf{u}, z, c).$$

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2)$.

Proof: (i) As we know that the trilinear form $b(\mathbf{u}, \mathbf{v}, \mathbf{v})$ can be expressed as

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \sum_{i,j=1}^2 \int_{\Omega} u_i (\partial_i v_j) v_j d\mathbf{x}.$$

Consider the following integral now

$$\int_{\Omega} u_i (\partial_i v_j) v_j d\mathbf{x} = \int_{\Omega} u_i \partial_i \left(\frac{v_j^2}{2} \right) d\mathbf{x}.$$

Using divergence theorem and as $\mathbf{u} = 0$ on Γ , therefore we have

$$\int_{\Omega} u_i (\partial_i v_j) v_j d\mathbf{x} = -\frac{1}{2} \int_{\Omega} \partial_i u_i (v_j^2) d\mathbf{x}.$$

Therefor we have

$$\sum_{i,j=1}^2 \int_{\Omega} u_i (\partial_i v_j) v_j d\mathbf{x} = -\frac{1}{2} \sum_{j=1}^2 \int_{\Omega} \operatorname{div}(\mathbf{u}) (v_j^2) d\mathbf{x}.$$

Since $\mathbf{u} \in V$, therefore we obtain the required result

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0.$$

The proofs of other two properties can be obtained by using similar arguments.

(ii) If we replace \mathbf{v} by $\mathbf{v} + \mathbf{w}$ in the first property of (i), we obtain

$$\begin{aligned} b(\mathbf{u}, \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) &= b(\mathbf{u}, \mathbf{v}, \mathbf{v}) + b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \\ &\quad + b(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b(\mathbf{u}, \mathbf{w}, \mathbf{w}). \end{aligned}$$

Making use of result (i), we get

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}).$$

Similarly we can prove the other two properties. □

Lemma 3 *Let assumptions (H1) – (H5) are satisfied, then for sufficiently regular (\mathbf{u}, ψ, c) we have for all $i = 2, 3$*

$$(i) \quad |\nabla \mathbf{A}_1(\mathbf{x}, t, \psi, c)|_2 \leq C (1 + |\nabla \psi|_2 + |\nabla c|_2),$$

$$(ii) \quad |\nabla A_i(\mathbf{x}, t, \psi, c)|_2 \leq C (1 + |\nabla \psi|_2 + |\nabla c|_2),$$

$$(iii) \quad |\nabla D(\mathbf{x}, t, \psi)|_2 \leq C (1 + |\nabla \psi|_2),$$

$$(iv) \quad |\nabla b(\mathbf{x}, t, \psi)|_2 \leq C (1 + |\nabla \psi|_2).$$

Proof: As we can write

$$\nabla (\mathbf{A}_1(\mathbf{x}, t, \psi, c)) = \mathbf{A}'_{1\mathbf{x}}(\mathbf{x}, t, \psi, c) + \mathbf{A}'_{1\psi}(\mathbf{x}, t, \psi, c) \nabla \psi + \mathbf{A}'_{1c}(\mathbf{x}, t, \psi, c) \nabla c,$$

by taking absolute on both sides, we have

$$\begin{aligned} |\nabla (\mathbf{A}_1(\mathbf{x}, t, \mathbf{u}, \psi, c))|_2 &\leq |\mathbf{A}'_{1\mathbf{x}}(\mathbf{x}, t, \psi, c)|_2 + |\mathbf{A}'_{1\psi}(\mathbf{x}, t, \psi, c)|_2 |\nabla \psi|_2 \\ &\quad + |\mathbf{A}'_{1c}(\mathbf{x}, t, \psi, c)|_2 |\nabla c|_2, \end{aligned}$$

since $\mathbf{A}_1 \in W^{1,\infty}$, therefore we have

$$|\nabla(\mathbf{A}_1(\mathbf{x}, t, \mathbf{u}, \psi, c))|_2 \leq C(1 + |\nabla\psi|_2 + |\nabla c|_2).$$

And by using the same technique, we can easily obtain the remaining three results (ii), (iii) and (iv) (see for similar results [6]). \square

Lemma 4 *Let the hypothesis (H1) – (H5) are satisfied and $\mathbf{X}_{m,n} = (\mathbf{u}_{m,n}, \psi_{m,n}, c_{m,n})$ be a sequence converging to $\mathbf{X} = (\mathbf{u}, \psi, c)$ in $L^2(0, T_f; \mathbf{H})$ strongly and in $L^2(0, T_f; \mathbf{V})$ weakly. Then we have the following convergence results*

- (i) $\mathbf{A}_1(\psi_{m,n}, c_{m,n}) \rightarrow \mathbf{A}_1(\psi, c)$, in $L^p(\mathcal{Q})$ strongly $\forall p \in [1, \infty)$,
- (ii) $A_i(\psi_{m,n}, c_{m,n}) \rightarrow A_i(\psi, c)$, $i = 2, 3$, in $L^p(\mathcal{Q})$ strongly $\forall p \in [1, \infty)$,
- (iii) $D(\psi_{m,n}) \rightarrow D(\psi)$, in $L^p(\mathcal{Q})$ strongly $\forall p \in [1, \infty)$,
- (iv) $D(\psi_{m,n})\nabla c_{m,n} \rightharpoonup D(\psi)\nabla c$, in $L^p(\mathcal{Q})$ weakly $\forall p \in [1, 2)$,
- (v) $A_3(\psi_{m,n}, c_{m,n})\nabla\psi_{m,n} \rightharpoonup A_3(\psi, c)\nabla\psi$, in $L^p(\mathcal{Q})$ weakly $\forall p \in [1, 2)$,

Proof: The proofs of the first three parts and the last two parts of the this lemma are similar, therefore we provide only the proofs of first and last part.

(i) Let $\mathbf{v} \in L^q(\mathcal{Q})$, for $q \in (1, +\infty)$ and consider

$$\mathcal{I}_{m,n} = \int_{\mathcal{Q}} (\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)) \mathbf{v} d\mathbf{x}dt,$$

or we can write

$$|\mathcal{I}_{m,n}| \leq \int_{\mathcal{Q}} |\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)|_2 |\mathbf{v}|_2 d\mathbf{x}dt,$$

and by using Hölder's inequality, we arrive at

$$|\mathcal{I}_{m,n}| \leq \|\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)\|_{L^p(\mathcal{Q})} \|\mathbf{v}\|_{L^q(\mathcal{Q})},$$

with $1/p + 1/q = 1$ and $1 \leq p < \infty$. Consider now

$$\begin{aligned} \|\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)\|_{L^p(\mathcal{Q})}^p &= \int_{\mathcal{Q}} |\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)|_2^p d\mathbf{x}dt, \\ &= \int_{\mathcal{Q}} |\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)|_2^{p-1} |\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)|_2 d\mathbf{x}dt. \end{aligned}$$

As we know that \mathbf{A}_1 is bounded (see hypothesis (H1) – (H5)), therefore we have

$$\|\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)\|_{L^p(\mathcal{Q})}^p \leq 2a_1 \int_{\mathcal{Q}} |\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)|_2 d\mathbf{x}dt.$$

and also \mathbf{A}_1 is Lipschitz function, therefore we can write

$$\|\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)\|_{L^p(\mathcal{Q})}^p \leq c_1 \left(\int_{\mathcal{Q}} (|\psi_{m,n} - \psi|^2 + |c_{m,n} - c|^2) d\mathbf{x}dt \right)^{1/2}.$$

The above inequality can be written as

$$\|\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)\|_{L^p(\mathcal{Q})}^p \leq c_2 \left(\|\psi_{m,n} - \psi\|_{L^2(\mathcal{Q})} + \|c_{m,n} - c\|_{L^2(\mathcal{Q})} \right).$$

As $\mathbf{X}_{m,n} = (\mathbf{u}_{m,n}, \psi_{m,n}, c_{m,n})$ converges strongly to $\mathbf{X} = (\mathbf{u}, \psi, c)$ in $L^2(0, T_f; \mathbf{H})$, therefore

$$\|\mathbf{A}_1(\psi_{m,n}, c_{m,n}) - \mathbf{A}_1(\psi, c)\|_{L^p(\mathcal{Q})}^p \longrightarrow 0, \text{ as } m, n \rightarrow \infty.$$

and consequently

$$|\mathcal{I}_{m,n}| \longrightarrow 0, \text{ as } m, n \rightarrow \infty.$$

This proves the result.

(v) Let $\phi \in L^q(\mathcal{Q})$ with $1/p + 1/q = 1$, $p \in [1, 2)$ and consider

$$\mathcal{K}_{m,n} = \int_{\mathcal{Q}} (A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} - A_3(\psi, c) \nabla \psi) \phi d\mathbf{x}dt,$$

adding and subtracting the term $A_3(\psi, c) \nabla \psi_{m,n}$, we obtain

$$\begin{aligned} \mathcal{K}_{m,n} &= \int_{\mathcal{Q}} (A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)) \nabla \psi_{m,n} \phi d\mathbf{x}dt \\ &\quad + \int_{\mathcal{Q}} (\nabla \psi_{m,n} - \nabla \psi) A_3(\psi, c) \phi d\mathbf{x}dt. \end{aligned}$$

As $\psi_{m,n}$ converges weakly to ψ in $L^2(0, T_f, H^1(\Omega))$, therefore second term in the above expression tends to zero as $m, n \rightarrow \infty$. Let $\mathcal{K}'_{m,n}$ denotes the first term in the above expression

$$\mathcal{K}'_{m,n} = \int_{\mathcal{Q}} (A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)) \nabla \psi_{m,n} \phi d\mathbf{x}dt,$$

or

$$|\mathcal{K}'_{m,n}| \leq \int_{\mathcal{Q}} |A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)| |\nabla \psi_{m,n}|_2 |\phi| d\mathbf{x}dt,$$

and using Hölder's inequality, we have

$$|\mathcal{K}'_{m,n}| \leq \|A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)\|_{L^s(\mathcal{Q})} \|\nabla \psi_{m,n}\|_{L^2(\mathcal{Q})} \|\phi\|_{L^q(\mathcal{Q})},$$

with $1/s + 1/2 + 1/q = 1$ and $s \geq 2$, and

$$\begin{aligned} \|A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)\|_{L^s(\mathcal{Q})}^s &= \int_{\mathcal{Q}} |A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)|_2^s d\mathbf{x}dt \\ &= \int_{\mathcal{Q}} |A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)|_2^{s-2} |A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)|_2^2 d\mathbf{x}dt. \end{aligned}$$

Since A_3 is bounded function (see hypothesis (H1) – (H5)), therefore we have

$$\|A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)\|_{L^s(\mathcal{Q})}^s \leq 2a_3 \int_{\mathcal{Q}} |A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)|_2^2 d\mathbf{x}dt,$$

also as A_3 is Lipschitz function, therefore we can write

$$\|A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)\|_{L^s(\mathcal{Q})}^s \leq 2c_0 \int_{\mathcal{Q}} (|\psi_{m,n} - \psi|_2^2 + |c_{m,n} - c|_2^2) d\mathbf{x}dt.$$

The above inequality can be written as

$$\|A_3(\psi_{m,n}, c_{m,n}) - A_3(\psi, c)\|_{L^s(\mathcal{Q})}^s \leq 2c_0 \left(\|\psi_{m,n} - \psi\|_{L^2(\mathcal{Q})}^2 + \|c_{m,n} - c\|_{L^2(\mathcal{Q})}^2 \right).$$

As $\mathbf{X}_{m,n} = (\mathbf{u}_{m,n}, \psi_{m,n}, c_{m,n})$ converges to $\mathbf{X} = (\mathbf{u}, \psi, c)$ in $L^2(0, T_f, \mathbf{H})$ strongly and weakly in $L^2(0, T_f, \mathbf{V})$, therefore $\mathcal{K}'_{m,n} \rightarrow 0$ as $m, n \rightarrow \infty$ and hence

$$|\mathcal{K}_{m,n}| \longrightarrow 0, \quad \text{as } m, n \rightarrow \infty.$$

This completes the proof. □

2.5 Existence and Regularity of the Solution

Theorem 1 *Let the assumptions (H1) – (H5) are satisfied and $(\mathbf{u}_0, \psi_0, c_0) \in \mathbf{H}$, then there exists a triplet (\mathbf{u}, ψ, c) such that*

$$\begin{aligned} (\mathbf{u}, \psi, c) &\in L^\infty(0, T_f, \mathbf{H}) \cap L^2(0, T_f, \mathbf{V}), \\ \left(\frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \psi}{\partial t}, \frac{\partial c}{\partial t} \right) &\in L^2(0, T_f, \mathbf{V}'), \end{aligned}$$

which is the solution of the problem (2.11).

Proof: We shall employ the Bubnov-Galerkin method to prove the existence of the problem (2.11). We shall approximate the system equations by projecting them onto finite dimensional subspaces. Since the boundary data in the problem (1.103) for velocity is different from the boundary data for the phase-field and concentration

equations, therefore we have to project the velocity equation onto m dimensional subspace and the phase-field and concentration equation onto the n dimensional subspace, then take the limit first in n and then in m . For this consider a sequence $(\lambda_i)_{i \geq 1}$ of the eigenvalues of the self adjoint operator $-\Delta$ with the homogeneous Dirichlet boundary conditions such that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots$ and the corresponding eigenfunctions $(w_i)_{i \geq 1}$, that satisfies

$$\begin{aligned} -\Delta \mathbf{w}_i &= \lambda_i \mathbf{w}_i \quad \text{in } \left((H_0^1(\Omega))^2 \right)' \quad \text{and } \mathbf{w}_i \in (H_0^1(\Omega))^2, \\ (\nabla \mathbf{w}_i, \nabla \mathbf{w}_j) &= (-\Delta \mathbf{w}_i, \mathbf{w}_j) = \lambda_i (\mathbf{w}_i, \mathbf{w}_j) \quad \forall \mathbf{w}_i, \mathbf{w}_j \in (H_0^1(\Omega))^2. \end{aligned} \quad (2.12)$$

Also the eigen functions satisfy $(\mathbf{w}_i, \mathbf{w}_j) = \delta_{ij}$ and $(\nabla \mathbf{w}_i, \nabla \mathbf{w}_j) = \lambda_i \delta_{ij}$ for $i \neq j$, $i, j \geq 1$. Furthermore, the eigenfunctions are smooth functions and form complete orthogonal basis in both $(L^2(\Omega))^2$ and $(H_0^1(\Omega))^2$. We denote by \mathbf{V}_m , the finite dimensional vector space generated by the eigenfunctions $(\mathbf{w}_i)_{1 \leq i \leq m}$. Then the union $\bigcup_{m \geq 1} \mathbf{V}_m$ is dense in both $(L^2(\Omega))^2$ and $(H_0^1(\Omega))^2$.

Consider again a sequence $(\mu_k)_{k \geq 1}$ of the eigenvalues of the operator $-\Delta$ with the homogeneous Neumann boundary conditions such that $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_k \leq \dots$ and the corresponding eigenfunctions $(e_k)_{k \geq 1}$ that is

$$-\Delta e_k = \mu_k e_k \quad \text{in } (H^1(\Omega))' \quad \text{and } e_k \in H^1(\Omega). \quad (2.13)$$

The eigenfunctions $(e_k)_{k \geq 1}$ satisfy $(e_k, e_l) = \delta_{kl}$ and $(\nabla e_k, \nabla e_l) = 0$ for $k \neq l$, $k, l \geq 1$. Moreover the eigenfunctions are smooth functions and $(e_k)_{k \geq 1}$ form a complete orthogonal basis in both $L^2(\Omega)$ and $H^1(\Omega)$. Let W_n be the finite vector space generated by $(e_k)_{k \geq 1}$. Then $\bigcup_{n \geq 1} W_n$ is dense in $L^2(\Omega)$ and also in $H^1(\Omega)$.

Now we define the L^2 , H_0^1 and H^1 -orthogonal projectors on the spaces \mathbf{V}_m and W_n respectively. Let P_m be the L^2 -orthogonal projector onto the space \mathbf{V}_m , such that $\forall \mathbf{v} \in (L^2(\Omega))^2$

$$(P_m \mathbf{v} - \mathbf{v}, \mathbf{w})_{L^2(\Omega)} = 0, \quad \forall \mathbf{w} \in \mathbf{V}_m, \quad (2.14)$$

and P_m to be the H_0^1 -orthogonal projector on \mathbf{V}_m , we should have $\forall \mathbf{v} \in (H_0^1(\Omega))^2$ that

$$(\nabla(P_m \mathbf{v} - \mathbf{v}), \nabla \mathbf{w}) = 0, \quad \forall \mathbf{w} \in \mathbf{V}_m. \quad (2.15)$$

Let L_n be the L^2 -orthogonal projections onto the space W_n , such that $\forall \phi \in L^2(\Omega)$

$$(L_n \phi - \phi, \varphi)_{L^2(\Omega)} = 0, \quad \forall \varphi \in W_n, \quad (2.16)$$

and L_n to be the H^1 -orthogonal projector on W_n , we should have $\forall \phi \in H^1(\Omega)$ that

$$(\nabla(L_n \phi - \phi), \nabla \varphi) = 0, \quad \forall \varphi \in W_n. \quad (2.17)$$

Using the above relations we can easily prove the following relations. For all $\mathbf{v} \in (L^2(\Omega))^2$ and $\phi \in L^2(\Omega)$,

$$\left. \begin{aligned} \|P_m \mathbf{v}\|_{L^2(\Omega)} &\leq C \|\mathbf{v}\|_{L^2(\Omega)}, \\ \|L_n \phi\|_{L^2(\Omega)} &\leq C \|\phi\|_{L^2(\Omega)}. \end{aligned} \right\} \quad (2.18)$$

For $\mathbf{v} \in (H_0^1(\Omega))^2$,

$$\|\nabla(P_m \mathbf{v})\|_{L^2(\Omega)} \leq C \|\nabla \mathbf{v}\|_{L^2(\Omega)}, \quad (2.19)$$

and for all $\phi \in H^1(\Omega)$, we have

$$\|\nabla(L_n \phi)\|_{L^2(\Omega)} \leq C \|\nabla \phi\|_{L^2(\Omega)}. \quad (2.20)$$

Moreover, if $\mathbf{v} \in (H^2(\Omega))^2$ and $\phi \in H_0^2(\Omega)$, we can easily prove that

$$\left. \begin{aligned} \|\Delta(P_m \mathbf{v})\|_{L^2(\Omega)} &\leq C \|\Delta \mathbf{v}\|_{L^2(\Omega)}, \\ \|\Delta(L_n \phi)\|_{L^2(\Omega)} &\leq C \|\Delta \phi\|_{L^2(\Omega)}, \end{aligned} \right\} \quad (2.21)$$

where $C > 0$ is a constant which is independent of m and n .

Applying the projections P_m on the first and L_n on the second and third equations of the system (2.11) respectively, we have $\forall (\mathbf{w}_i, e_k) \in \mathbf{V}_m \times W_n$,

$$\begin{aligned} &\rho_0 \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{w}_i \right) + a_u(\mathbf{u}_{m,n}, \mathbf{w}_i) + b_u(\mathbf{u}_{m,n}; \mathbf{u}_{m,n}, \mathbf{w}_i) \\ &= \int_{\Omega} \mathbf{A}_1(\psi_{m,n}, c_{m,n}) \cdot \mathbf{w}_i \, d\mathbf{x} + \int_{\Omega} b(\psi)((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{w}_i \, d\mathbf{x}, \end{aligned} \quad (2.22)$$

$$\begin{aligned} &\left(\frac{\partial \psi_{m,n}}{\partial t}, e_k \right) + a_\psi(\psi_{m,n}, e_k) + b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, e_k) \\ &= - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) e_k \, d\mathbf{x}, \end{aligned} \quad (2.23)$$

$$\begin{aligned} &\left(\frac{\partial c_{m,n}}{\partial t}, e_k \right) + b_c(\mathbf{u}_{m,n}, c_{m,n}, e_k) + \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla e_k \, d\mathbf{x} \\ &+ \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla e_k \, d\mathbf{x} = 0, \end{aligned} \quad (2.24)$$

$$(\mathbf{u}_{m,n}, \psi_{m,n}, c_{m,n})(t=0) = (\mathbf{u}_0^{m,n}, \psi_0^{m,n}, c_0^{m,n}), \quad (2.25)$$

where $(P_m \mathbf{u}_0, L_n \psi_0, L_n c_0) = (\mathbf{u}_0^{m,n}, \psi_0^{m,n}, c_0^{m,n})$ which satisfy

$$(\mathbf{u}_0^{m,n}, \psi_0^{m,n}, c_0^{m,n}) \rightarrow (\mathbf{u}_0, \psi_0, c_0) \quad \text{in } \mathbf{H} \quad \text{as } m, n \rightarrow \infty, \quad (2.26)$$

where the Bubnov-Galerkin approximation can be given for each $m, n \geq 1$, as

$$\left. \begin{aligned} \mathbf{u}_{m,n}(\cdot, t) &= \sum_{i=1}^m u_i^{m,n}(t) \mathbf{w}_i, \\ \psi_{m,n}(\cdot, t) &= \sum_{k=1}^n \psi_k^{m,n}(t) e_k, \\ c_{m,n}(\cdot, t) &= \sum_{k=1}^n c_k^{m,n}(t) e_k. \end{aligned} \right\} \quad (2.27)$$

Now multiplying equations (2.22), (2.23) and (2.24) respectively by $u_i^{m,n}(t)$, $\psi_k^{m,n}(t)$ and $c_k^{m,n}(t)$ and then taking sum over i and k , where $i = 1$ to m , $k = 1$ to n , we obtain

$$\begin{aligned} & \rho_0 \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{u}_{m,n} \right) + a_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}) + b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{u}_{m,n}) \\ &= \int_{\Omega} \mathbf{A}_1(\psi_{m,n}, c_{m,n}) \cdot \mathbf{u}_{m,n} \, d\mathbf{x} + \int_{\Omega} b(\psi)((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{u}_{m,n} \, d\mathbf{x}, \\ & \left(\frac{\partial \psi_{m,n}}{\partial t}, \psi_{m,n} \right) + a_{\psi}(\psi_{m,n}, \psi_{m,n}) + b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \psi_{m,n}) \\ &= - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \psi_{m,n} \, d\mathbf{x}, \\ & \left(\frac{\partial c_{m,n}}{\partial t}, c_{m,n} \right) + b_c(\mathbf{u}_{m,n}, c_{m,n}, c_{m,n}) + \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla c_{m,n} \, d\mathbf{x} \\ &+ \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla c_{m,n} \, d\mathbf{x} = 0. \end{aligned}$$

Making use of Lemma (2), the above equations takes the form

$$\begin{aligned} & \rho_0 \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{u}_{m,n} \right) + a_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}) = \int_{\Omega} \mathbf{A}_1(\psi_{m,n}, c_{m,n}) \cdot \mathbf{u}_{m,n} \, d\mathbf{x} \\ &+ \int_{\Omega} b(\psi)((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{u}_{m,n} \, d\mathbf{x}, \\ & \left(\frac{\partial \psi_{m,n}}{\partial t}, \psi_{m,n} \right) + a_{\psi}(\psi_{m,n}, \psi_{m,n}) = - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \psi_{m,n} \, d\mathbf{x}, \\ & \left(\frac{\partial c_{m,n}}{\partial t}, c_{m,n} \right) + \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla c_{m,n} \, d\mathbf{x} \\ &+ \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla c_{m,n} \, d\mathbf{x} = 0. \end{aligned}$$

The above equations can further be written as

$$\begin{aligned} \frac{\rho_0}{2} \frac{d}{dt} |\mathbf{u}_{m,n}|^2 + \mu |\nabla \mathbf{u}_{m,n}|^2 &= \int_{\Omega} \mathbf{A}_1(\psi_{m,n}, c_{m,n}) \cdot \mathbf{u}_{m,n} \, d\mathbf{x} \\ &+ \int_{\Omega} b(\psi)((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{u}_{m,n} \, d\mathbf{x}, \end{aligned} \quad (2.28)$$

$$\frac{1}{2} \frac{d}{dt} |\psi_{m,n}|^2 + \epsilon_1 |\nabla \psi_{m,n}|^2 = - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \psi_{m,n} \, d\mathbf{x}, \quad (2.29)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |c_{m,n}|^2 + \int_{\Omega} D(\psi_{m,n}) |\nabla c_{m,n}|_2^2 \, d\mathbf{x} \\ = - \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla c_{m,n} \, d\mathbf{x}, \end{aligned} \quad (2.30)$$

and by using the hypothesis (H1) – (H5), in the above equations, we obtain

$$\begin{aligned} \frac{\rho_0}{2} \frac{d}{dt} |\mathbf{u}_{m,n}|^2 + \mu |\nabla \mathbf{u}_{m,n}|^2 &\leq a_1 \int_{\Omega} |\mathbf{u}_{m,n}|_2 \, d\mathbf{x} + b_1 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\mathbf{u}_{m,n}|_2 \, d\mathbf{x}, \\ \frac{1}{2} \frac{d}{dt} |\psi_{m,n}|^2 + \epsilon_1 |\nabla \psi_{m,n}|^2 &\leq a_2 \int_{\Omega} |\psi_{m,n}| \, d\mathbf{x}, \\ \frac{1}{2} \frac{d}{dt} |c_{m,n}|^2 + D_0 \int_{\Omega} |\nabla c_{m,n}|_2^2 \, d\mathbf{x} &\leq a_3 \int_{\Omega} |\nabla \psi_{m,n}|_2 |\nabla c_{m,n}|_2 \, d\mathbf{x}. \end{aligned}$$

Using Hölder's inequality, the above inequalities take the form

$$\begin{aligned} \rho_0 \frac{d}{dt} |\mathbf{u}_{m,n}|^2 + 2\mu |\nabla \mathbf{u}_{m,n}|^2 &\leq c_1 |\mathbf{u}_{m,n}|^2 + c_2 |\mathbf{u}_{m,n}|, \\ \frac{d}{dt} |\psi_{m,n}|^2 + 2\epsilon_1 |\nabla \psi_{m,n}|^2 &\leq c_3 |\psi_{m,n}|, \\ \frac{d}{dt} |c_{m,n}|^2 + 2D_0 |\nabla c_{m,n}|^2 &\leq c_4 |\nabla \psi_{m,n}| |\nabla c_{m,n}|. \end{aligned}$$

Further by Young's inequality, we have for $\delta_1 > 0$

$$\rho_0 \frac{d}{dt} |\mathbf{u}_{m,n}|^2 + 2\mu |\nabla \mathbf{u}_{m,n}|^2 \leq c_5 + c_6 |\mathbf{u}_{m,n}|^2, \quad (2.31)$$

$$\frac{d}{dt} |\psi_{m,n}|^2 + 2\epsilon_1 |\nabla \psi_{m,n}|^2 \leq c_7 + |\psi_{m,n}|^2, \quad (2.32)$$

$$\frac{d}{dt} |c_{m,n}|^2 + 2D_0 |\nabla c_{m,n}|^2 \leq \frac{c_4^2}{4\delta_1} |\nabla \psi_{m,n}|^2 + \delta_1 |\nabla c_{m,n}|^2. \quad (2.33)$$

Multiplying the inequality (2.33) by δ_2 on both sides, we get

$$\delta_2 \frac{d}{dt} |c_{m,n}|^2 + 2\delta_2 D_0 |\nabla c_{m,n}|^2 \leq \frac{c_4^2 \delta_2}{4\delta_1} |\nabla \psi_{m,n}|^2 + \delta_2 \delta_1 |\nabla c_{m,n}|^2,$$

choosing $\delta_1 = D_0$ and $\delta_2 = \frac{4\epsilon_1 D_0}{c_4^2}$, the above inequality takes the form

$$\delta_2 \frac{d}{dt} |c_{m,n}|^2 + 2\delta_2 D_0 |\nabla c_{m,n}|^2 \leq \epsilon_1 |\nabla \psi_{m,n}|^2 + \delta_2 D_0 |\nabla c_{m,n}|^2. \quad (2.34)$$

Adding the inequalities (2.31), (2.32) and (2.34), we obtain

$$\begin{aligned} \frac{d}{dt} (\rho_0 |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + \delta_2 |c_{m,n}|^2) + 2\mu |\nabla \mathbf{u}_{m,n}|^2 + \epsilon_1 |\nabla \psi_{m,n}|^2 \\ + \delta_2 D_0 |\nabla c_{m,n}|^2 \leq c_8 + c_6 |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2, \end{aligned}$$

and since $|c_{m,n}|^2 \geq 0$, therefore the above inequality takes the form

$$\begin{aligned} \frac{d}{dt} (\rho_0 |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + \delta_2 |c_{m,n}|^2) + 2\mu |\nabla \mathbf{u}_{m,n}|^2 + \epsilon_1 |\nabla \psi_{m,n}|^2 \\ + \delta_2 D_0 |\nabla c_{m,n}|^2 \leq c_9 (1 + |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + |c_{m,n}|^2). \end{aligned} \quad (2.35)$$

From the above relation, we can deduce that

$$\frac{d}{dt} (\rho_0 |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + \delta_2 |c_{m,n}|^2) \leq c_9 (1 + |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + |c_{m,n}|^2).$$

Let $c_{10} = \min(\rho_0, 1, \delta_2)$ and $c_{11} = c_9/c_{10}$, then we have

$$\frac{d}{dt} (|\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + |c_{m,n}|^2) \leq c_{11} (1 + |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + |c_{m,n}|^2).$$

Using Gronwall's lemma, the inequality (2.18), the equation (2.26) and as $(\mathbf{u}_0, \psi_0, c_0) \in \mathbf{H}$, therefore we have

$$|\mathbf{u}_{m,n}(t)|^2 + |\psi_{m,n}(t)|^2 + |c_{m,n}(t)|^2 \leq c_{12}, \quad \forall t \in (0, T_f).$$

We can simplify the above expression as

$$|\mathbf{X}_{m,n}(t)|^2 \leq c_{12}, \quad \forall t \in (0, T_f),$$

where $\mathbf{X}_{m,n} = (\mathbf{u}_{m,n}, \psi_{m,n}, c_{m,n})$. Thus we conclude that

$$\|\mathbf{X}_{m,n}\|_{L^\infty(0, T_f; \mathbf{H})} \leq c_{12}. \quad (2.36)$$

This implies that $\mathbf{X}_{m,n} = (\mathbf{u}_{m,n}, \psi_{m,n}, c_{m,n})$ is uniformly bounded in $L^\infty(0, T_f; \mathbf{H})$.

Now integrating equation (2.35) over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\begin{aligned} \int_0^t \frac{d}{ds} (\rho_0 |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + \delta_2 |c_{m,n}|^2) ds + 2\mu \int_0^t |\nabla \mathbf{u}_{m,n}|^2 ds \\ + \epsilon_1 \int_0^t |\nabla \psi_{m,n}|^2 ds + \delta_2 D_0 \int_0^t |\nabla c_{m,n}|^2 ds \\ \leq c_9 \int_0^t (1 + |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + |c_{m,n}|^2) ds, \end{aligned}$$

or we can write the above inequality as

$$\begin{aligned}
& (\rho_0 |\mathbf{u}_{m,n}(t)|^2 + |\psi_{m,n}(t)|^2 + \delta_2 |c_{m,n}(t)|^2) + 2\mu \int_0^t |\nabla \mathbf{u}_{m,n}|^2 ds \\
& \quad + \epsilon_1 \int_0^t |\nabla \psi_{m,n}|^2 ds + \delta_2 D_0 \int_0^t |\nabla c_{m,n}|^2 ds \\
& \leq c_9 \int_0^t (1 + |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + |c_{m,n}|^2) ds \\
& \quad + (\rho_0 |\mathbf{u}_{m,n}(0)|^2 + |\psi_{m,n}(0)|^2 + \delta_2 |c_{m,n}(0)|^2).
\end{aligned}$$

From above inequality, we can deduce that

$$\begin{aligned}
& 2\mu \int_0^t |\nabla \mathbf{u}_{m,n}|^2 ds + \epsilon_1 \int_0^t |\nabla \psi_{m,n}|^2 ds + \delta_2 D_0 \int_0^t |\nabla c_{m,n}|^2 ds \\
& \leq c_9 \int_0^t (1 + |\mathbf{u}_{m,n}|^2 + |\psi_{m,n}|^2 + |c_{m,n}|^2) ds \\
& \quad + c_{13} (\rho_0 |\mathbf{u}_0|^2 + |\psi_0|^2 + \delta_2 |c_0|^2),
\end{aligned}$$

and according to (2.18), (2.36) and as $(\mathbf{u}_0, \psi_0, c_0) \in \mathbf{H}$, we have

$$\int_0^t (2\mu |\nabla \mathbf{u}_{m,n}|^2 + \epsilon_1 |\nabla \psi_{m,n}|^2 + \delta_2 D_0 |\nabla c_{m,n}|^2) ds \leq c_{14}, \quad \forall t \in (0, T_f)$$

or

$$\min(2\mu, \epsilon_1, \delta_2 D_0) \int_0^{T_f} (|\nabla \mathbf{u}_{m,n}|^2 + |\nabla \psi_{m,n}|^2 + |\nabla c_{m,n}|^2) dt \leq c_{15}.$$

We can write above inequality as

$$\int_0^{T_f} |\nabla \mathbf{X}_{m,n}|^2 dt \leq c_{16}.$$

Therefore we have the following result

$$\|\nabla \mathbf{X}_{m,n}\|_{L^2(0, T_f; \mathbf{H})}^2 \leq c_{16}. \quad (2.37)$$

Also as we know that

$$\|\mathbf{X}_{m,n}\|^2 = |\mathbf{X}_{m,n}|^2 + |\nabla \mathbf{X}_{m,n}|^2.$$

Integrating above equation over $(0, T_f)$ and using the results (2.36) and (2.37), we conclude that

$$\int_0^{T_f} \|\mathbf{X}_{m,n}\|^2 dt \leq c_{17}.$$

Thus we have

$$\|\mathbf{X}_{m,n}\|_{L^2(0, T_f; \mathbf{V})} \leq c_{17}. \quad (2.38)$$

This implies that $\mathbf{X}_{m,n}$ is uniformly bounded in $L^2(0, T_f; \mathbf{V})$. From the results (2.36) and (2.38), we conclude that $\mathbf{X}_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; \mathbf{H}) \cap L^2(0, T_f; \mathbf{V})$. That is

$$\|\mathbf{X}_{m,n}\|_{L^\infty(0, T_f; \mathbf{H}) \cap L^2(0, T_f; \mathbf{V})} \leq c_{18}. \quad (2.39)$$

This result makes it possible to extract a subsequence from $\mathbf{X}_{m,n}$, also denoted by $\mathbf{X}_{m,n}$, which converges weak star toward $\mathbf{X} = (\mathbf{u}, \psi, c)$ in $L^\infty(0, T_f; \mathbf{H})$ and converges weakly toward \mathbf{X} in $L^2(0, T_f; \mathbf{V})$.

Now from equations (2.22)-(2.24), we have $\forall (\mathbf{v}, \phi, z) \in \mathbf{V}_m \times W_n \times W_n$,

$$\begin{aligned} \rho_0 \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{v} \right) &= -\mu \int_{\Omega} \nabla \mathbf{u}_{m,n} \cdot \nabla \mathbf{v} d\mathbf{x} - b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{v}) \\ &\quad + (\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \mathbf{v}) + (b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \mathbf{v}), \\ \left(\frac{\partial \psi_{m,n}}{\partial t}, \phi \right) &= -b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \phi) - \epsilon_1 \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \phi d\mathbf{x} - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \phi d\mathbf{x}, \\ \left(\frac{\partial c_{m,n}}{\partial t}, z \right) &= -b_c(\mathbf{u}_{m,n}, c_{m,n}, z) - \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla z d\mathbf{x} \\ &\quad - \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla z d\mathbf{x}, \end{aligned}$$

and using hypothesis (H1) – (H5), we arrive at

$$\begin{aligned} \rho_0 \left| \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{v} \right) \right| &\leq \mu \int_{\Omega} |\nabla \mathbf{u}_{m,n}|_2 |\nabla \mathbf{v}|_2 d\mathbf{x} + |b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{v})| \\ &\quad + a_1 \int_{\Omega} |\mathbf{v}|_2 d\mathbf{x} + b_1 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\mathbf{v}|_2 d\mathbf{x}, \\ \left| \left(\frac{\partial \psi_{m,n}}{\partial t}, \phi \right) \right| &\leq |b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \phi)| + \epsilon_1 \int_{\Omega} |\nabla \psi_{m,n}|_2 |\nabla \phi|_2 d\mathbf{x} + a_2 \int_{\Omega} |\phi|_2 d\mathbf{x}, \\ \left| \left(\frac{\partial c_{m,n}}{\partial t}, z \right) \right| &\leq |b_c(\mathbf{u}_{m,n}, c_{m,n}, z)| + D_1 \int_{\Omega} |\nabla c_{m,n}|_2 |\nabla z|_2 d\mathbf{x} \\ &\quad + a_3 \int_{\Omega} |\nabla \psi_{m,n}|_2 |\nabla z|_2 d\mathbf{x}. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} \rho_0 \left| \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{v} \right) \right| &\leq \mu |\nabla \mathbf{u}_{m,n}| |\nabla \mathbf{v}| + |b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{v})| \\ &\quad + c_{18} |\mathbf{v}| + b_1 |\mathbf{u}_{m,n}| |\mathbf{v}|, \\ \left| \left(\frac{\partial \psi_{m,n}}{\partial t}, \phi \right) \right| &\leq |b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \phi)| + \epsilon_1 |\nabla \psi_{m,n}| |\nabla \phi| + c_{19} |\phi|, \end{aligned}$$

$$\left| \left(\frac{\partial c_{m,n}}{\partial t}, z \right) \right| \leq |b_c(\mathbf{u}_{m,n}, c_{m,n}, z)| + D_1 |\nabla c_{m,n}| |\nabla z| \\ + a_3 |\nabla \psi_{m,n}| |\nabla z|,$$

As $|\mathbf{v}| \leq \|\mathbf{v}\|$, $\forall \mathbf{v} \in V$ and $|\phi| \leq \|\phi\|$, $|\nabla \phi| \leq \|\phi\|$, $\forall \phi \in H^1(\Omega)$ by the definition of Sobolev norm. The above inequalities take the form

$$\rho_0 \left| \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{v} \right) \right| \leq \mu |\nabla \mathbf{u}_{m,n}| \|\mathbf{v}\| + |b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{v})| \\ + c_{18} \|\mathbf{v}\| + b_1 |\mathbf{u}_{m,n}| \|\mathbf{v}\|, \quad (2.40)$$

$$\left| \left(\frac{\partial \psi_{m,n}}{\partial t}, \phi \right) \right| \leq |b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \phi)| + \epsilon_1 |\nabla \psi_{m,n}| \|\phi\| + c_{19} \|\phi\|, \quad (2.41)$$

$$\left| \left(\frac{\partial c_{m,n}}{\partial t}, z \right) \right| \leq |b_c(\mathbf{u}_{m,n}, c_{m,n}, z)| + D_1 |\nabla c_{m,n}| \|z\| \\ + a_3 |\nabla \psi_{m,n}| \|z\|. \quad (2.42)$$

Now as (according to Lemma (2))

$$|b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{v})| = |b_u(\mathbf{u}_{m,n}, \mathbf{v}, \mathbf{u}_{m,n})|, \\ \leq \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\nabla \mathbf{v}|_2 |\mathbf{u}_{m,n}|_2 d\mathbf{x}.$$

By employing the Hölder's inequality and then Gagliardo-Nirenberg's inequality, the above inequality takes the form

$$|b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{v})| \leq \|\mathbf{u}_{m,n}\|_{\mathbf{L}^4(\Omega)}^2 |\nabla \mathbf{v}| \leq c_{20} \|\mathbf{u}_{m,n}\| |\mathbf{u}_{m,n}| \|\mathbf{v}\|,$$

and using the inequality (2.36), we obtain

$$|b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \mathbf{v})| \leq c_{21} \|\mathbf{u}_{m,n}\| \|\mathbf{v}\|. \quad (2.43)$$

Similarly we can derive that

$$|b_c(\mathbf{u}_{m,n}, c_{m,n}, z)| \leq c_{22} \|\mathbf{u}_{m,n}\|^{1/2} \|c_{m,n}\|^{1/2} \|z\|. \quad (2.44)$$

and

$$|b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \phi)| \leq c_{23} \|\mathbf{u}_{m,n}\|^{1/2} \|\psi_{m,n}\|^{1/2} \|\phi\|. \quad (2.45)$$

Using the results (2.43), (2.44) and (2.45) in the inequalities (2.40)-(2.42), we arrive at

$$\rho_0 \left| \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{v} \right) \right| \leq \mu |\nabla \mathbf{u}_{m,n}| \|\mathbf{v}\| + c_{21} \|\mathbf{u}_{m,n}\| \|\mathbf{v}\| \\ + c_{18} \|\mathbf{v}\| + b_1 |\mathbf{u}_{m,n}| \|\mathbf{v}\|,$$

$$\left| \left(\frac{\partial \psi_{m,n}}{\partial t}, \phi \right) \right| \leq c_{23} \|\mathbf{u}_{m,n}\|^{1/2} \|\psi_{m,n}\|^{1/2} \|\phi\| + \epsilon_1 |\nabla \psi_{m,n}| \|\phi\| + c_{19} \|\phi\|,$$

$$\left| \left(\frac{\partial c_{m,n}}{\partial t}, z \right) \right| \leq c_{22} \|\mathbf{u}_{m,n}\|^{1/2} \|c_{m,n}\|^{1/2} \|z\| + D_1 |\nabla c_{m,n}| \|z\| + a_3 |\nabla \psi_{m,n}| \|z\|,$$

the above inequalities can be written as

$$\rho_0 \left| \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \mathbf{v} \right) \right| \leq (\mu |\nabla \mathbf{u}_{m,n}| + c_{21} \|\mathbf{u}_{m,n}\| + c_{18} + b_1 |\mathbf{u}_{m,n}|) \|\mathbf{v}\|,$$

$$\left| \left(\frac{\partial \psi_{m,n}}{\partial t}, \phi \right) \right| \leq (c_{23} \|\mathbf{u}_{m,n}\|^{1/2} \|\psi_{m,n}\|^{1/2} + \epsilon_1 |\nabla \psi_{m,n}| + c_{19}) \|\phi\|,$$

$$\left| \left(\frac{\partial c_{m,n}}{\partial t}, z \right) \right| \leq (c_{22} \|\mathbf{u}_{m,n}\|^{1/2} \|c_{m,n}\|^{1/2} + D_1 |\nabla c_{m,n}| + a_3 |\nabla \psi_{m,n}|) \|z\|.$$

Dividing above set of relations respectively by $\|\mathbf{v}\|$, $\|\phi\|$ and $\|z\|$, we get

$$\rho_0 \left\| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right\|_{V'} \leq \mu |\nabla \mathbf{u}_{m,n}| + c_{21} \|\mathbf{u}_{m,n}\| + c_{18} + b_1 |\mathbf{u}_{m,n}|,$$

$$\left\| \frac{\partial \psi_{m,n}}{\partial t} \right\|_{(H^1(\Omega))'} \leq c_{23} \|\mathbf{u}_{m,n}\|^{1/2} \|\psi_{m,n}\|^{1/2} + \epsilon_1 |\nabla \psi_{m,n}| + c_{19},$$

$$\left\| \frac{\partial c_{m,n}}{\partial t} \right\|_{(H^1(\Omega))'} \leq c_{22} \|\mathbf{u}_{m,n}\|^{1/2} \|c_{m,n}\|^{1/2} + D_1 |\nabla c_{m,n}| + a_3 |\nabla \psi_{m,n}|,$$

where V' and $(H^1(\Omega))'$ are the corresponding dual spaces of V and $H^1(\Omega)$ respectively.

By employing Young's inequality, the above inequalities take the form

$$\rho_0 \left\| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right\|_{V'} \leq c_{24} (1 + \|\mathbf{u}_{m,n}\|),$$

$$\left\| \frac{\partial \psi_{m,n}}{\partial t} \right\|_{(H^1(\Omega))'} \leq c_{25} (1 + \|\mathbf{u}_{m,n}\| + \|\psi_{m,n}\|),$$

$$\left\| \frac{\partial c_{m,n}}{\partial t} \right\|_{(H^1(\Omega))'} \leq c_{26} (\|\mathbf{u}_{m,n}\| + \|c_{m,n}\| + |\nabla \psi_{m,n}|).$$

Making use of the result (2.39), we can easily arrive at

$$\rho_0 \int_0^{T_f} \left\| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right\|_{V'}^2 dt \leq c_{27},$$

$$\begin{aligned} \int_0^{T_f} \left\| \frac{\partial \psi_{m,n}}{\partial t} \right\|_{(H^1(\Omega))'}^2 dt &\leq c_{28}, \\ \int_0^{T_f} \left\| \frac{\partial c_{m,n}}{\partial t} \right\|_{(H^1(\Omega))'}^2 dt &\leq c_{29}. \end{aligned}$$

We can write

$$\int_0^{T_f} \left\| \frac{\partial \mathbf{X}_{m,n}}{\partial t} \right\|_{\mathbf{V}'}^2 dt \leq c_{30}.$$

This implies that

$$\left\| \frac{\partial \mathbf{X}_{m,n}}{\partial t} \right\|_{L^2(0, T_f; \mathbf{V}')} \leq c_{30}. \quad (2.46)$$

This shows that $\frac{\partial \mathbf{X}_{m,n}}{\partial t} = \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \frac{\partial \psi_{m,n}}{\partial t}, \frac{\partial c_{m,n}}{\partial t} \right)$ is uniformly bounded in $L^2(0, T_f; \mathbf{V}')$. Now we define the space

$$\mathcal{W}_1 = \left\{ \mathbf{X} \in L^2(0, T_f; \mathbf{V}), \frac{\partial \mathbf{X}}{\partial t} \in L^2(0, T_f; \mathbf{V}') \right\}.$$

According to [55], the injection of \mathcal{W}_1 into $L^2(0, T_f; \mathbf{H})$ is compact.

From equation (2.39) and (2.46), we conclude that $\mathbf{X}_{m,n}$ is uniformly bounded in \mathcal{W}_1 , therefore we can extract from $\mathbf{X}_{m,n}$, a subsequence also denoted by $\mathbf{X}_{m,n}$, such that

$$\begin{aligned} \mathbf{X}_{m,n} &\rightharpoonup \mathbf{X} \quad \text{weakly in } L^2(0, T_f; \mathbf{V}), \\ \mathbf{X}_{m,n} &\longrightarrow \mathbf{X} \quad \text{strongly in } L^2(0, T_f; \mathbf{H}), \end{aligned}$$

where $\mathbf{X}_{m,n} = (\mathbf{u}_{m,n}, \psi_{m,n}, c_{m,n})$ and $\mathbf{X} = (\mathbf{u}, \psi, c)$.

Now we shall prove that $\mathbf{X} = (\mathbf{u}, \psi, c)$ is the solution of the problem (2.11).

In order to pass limit in equations (2.22)-(2.25), we consider $\varphi \in C^1([0, T_f])$ such that $\varphi(T_f) = 0$.

Multiplying equations (2.22)-(2.24) by $\varphi(t)$ and then integrating with respect to t over $(0, T_f)$, we have $\forall (\mathbf{v}, \phi, z) \in \mathbf{V}_m \times W_n \times W_n$,

$$\begin{aligned} &\rho_0 \int_0^{T_f} \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \varphi \mathbf{v} \right) dt + \int_0^{T_f} a_u(\mathbf{u}_{m,n}, \varphi \mathbf{v}) dt + \int_0^{T_f} b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \varphi \mathbf{v}) dt \\ &= \int_0^{T_f} (\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \varphi \mathbf{v}) dt + \int_0^{T_f} (b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \varphi \mathbf{v}) dt, \end{aligned}$$

$$\begin{aligned} &\int_0^{T_f} \left(\frac{\partial \psi_{m,n}}{\partial t}, \varphi \phi \right) dt + \int_0^{T_f} a_\psi(\psi_{m,n}, \varphi \phi) dt \\ &= - \int_0^{T_f} \int_\Omega A_2(\psi_{m,n}, c_{m,n}) \varphi \phi \, d\mathbf{x} dt, \end{aligned}$$

$$\begin{aligned} & \int_0^{T_f} \left(\frac{\partial c_{m,n}}{\partial t}, \varphi z \right) dt + \int_0^{T_f} \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla(\varphi z) d\mathbf{x} dt \\ & + \int_0^{T_f} b_c(\mathbf{u}_{m,n}, c_{m,n}, \varphi z) dt + \int_0^{T_f} \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla(\varphi z) d\mathbf{x} dt = 0, \end{aligned}$$

and integrating the first terms of the above equations by parts with respect to t , and using $\varphi(T_f) = 0$, we have

$$\begin{aligned} & -\rho_0 \int_0^{T_f} (\mathbf{u}_{m,n}, \varphi' \mathbf{v}) dt + \int_0^{T_f} a_u(\mathbf{u}_{m,n}, \varphi \mathbf{v}) dt + \int_0^{T_f} b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \varphi \mathbf{v}) dt \\ & = \int_0^{T_f} (\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \varphi \mathbf{v}) dt + \int_0^{T_f} (b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \varphi \mathbf{v}) dt \\ & \quad + (\mathbf{u}_0^{m,n}, \varphi(0) \mathbf{v}), \\ & - \int_0^{T_f} (\psi_{m,n}, \varphi' \phi) dt + \int_0^{T_f} a_\psi(\psi_{m,n}, \varphi \phi) dt \\ & = - \int_0^{T_f} \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \varphi \phi d\mathbf{x} dt + (\psi_0^{m,n}, \varphi(0) \phi), \\ & - \int_0^{T_f} (c_{m,n}, \varphi' z) dt + \int_0^{T_f} \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla(\varphi z) d\mathbf{x} dt \\ & \quad + \int_0^{T_f} b_c(\mathbf{u}_{m,n}, c_{m,n}, \varphi z) dt + \int_0^{T_f} \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla(\varphi z) d\mathbf{x} dt \\ & = (c_0^{m,n}, \varphi(0) z). \end{aligned}$$

It is easy to pass the limit in the linear terms and for the nonlinear terms we use Lemma 4 and equation (2.26), therefore applying limit $(m, n \rightarrow \infty)$ in the above equations, we get

$$\begin{aligned} & -\rho_0 \int_0^{T_f} (\mathbf{u}, \varphi' \mathbf{v}) dt + \int_0^{T_f} a_u(\mathbf{u}, \varphi \mathbf{v}) dt + \int_0^{T_f} b_u(\mathbf{u}, \mathbf{u}, \varphi \mathbf{v}) dt \\ & = \int_0^{T_f} (\mathbf{A}_1(\psi, c), \varphi \mathbf{v}) dt + \int_0^{T_f} (b(\psi)((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}), \varphi \mathbf{v}) dt \\ & \quad + (\mathbf{u}_0, \varphi(0) \mathbf{v}), \end{aligned} \tag{2.47}$$

$$\begin{aligned} & - \int_0^{T_f} (\psi, \varphi' \phi) dt + \int_0^{T_f} a_\psi(\psi, \varphi \phi) dt \\ & = - \int_0^{T_f} \int_{\Omega} A_2(\psi, c) \varphi \phi d\mathbf{x} dt + (\psi_0, \varphi(0) \phi), \end{aligned} \tag{2.48}$$

$$\begin{aligned}
& - \int_0^{T_f} (c, \varphi' z) dt + \int_0^{T_f} \int_{\Omega} A_3(\psi, c) \nabla \psi \cdot \nabla (\varphi z) d\mathbf{x} dt \\
& \quad + \int_0^{T_f} b_c(\mathbf{u}, c, \varphi z) dt + \int_0^{T_f} \int_{\Omega} D(\psi) \nabla c \cdot \nabla (\varphi z) d\mathbf{x} dt \\
& \quad = (c_0, \varphi(0)z). \tag{2.49}
\end{aligned}$$

Assuming that $\varphi \in \mathcal{D}(0, T_f)$, we can deduce that (\mathbf{u}, ψ, c) verifies the problem (2.11) in the distribution sense on $(0, T_f)$.

Finally, it remains to verify the initial conditions. For this, multiplying each equation in problem (2.11) by $\varphi(t)$ and integrating with respect to t over $(0, T_f)$ and then comparing the resulting equations with the equations (2.47)-(2.49), we finally obtain that $(\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0)$. \square

Theorem 2 *Let $(\mathbf{u}_0, \psi_0, c_0) \in (H_0^1(\Omega))^2 \times H^1(\Omega) \times L^2(\Omega)$, and the assumptions (H1)–(H5) are satisfied. Then there exist a triplet of functions (\mathbf{u}, ψ, c) satisfying*

$$\begin{aligned}
\mathbf{u} & \in L^2\left(0, T_f; (H^2(\Omega))^2\right) \cap H^1\left(0, T_f; (L^2(\Omega))^2\right) \\
\psi & \in L^2(0, T_f; H^2(\Omega)) \cap H^1(0, T_f; L^2(\Omega)) \\
c & \in L^2(0, T_f; H^1(\Omega)) \cap L^2(0, T_f; (H^1(\Omega))')
\end{aligned}$$

which is a solution of the problem (2.11).

Proof: Multiplying equation (2.22) by λ_i and then using equation (2.12), we have

$$\begin{aligned}
& -\rho_0 \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \Delta \mathbf{w}_i \right) - \mu \int_{\Omega} \nabla \mathbf{u}_{m,n} \cdot \nabla (\Delta \mathbf{w}_i) d\mathbf{x} - b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{w}_i) \\
& = -(\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \Delta \mathbf{w}_i) - (b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \Delta \mathbf{w}_i), \tag{2.50}
\end{aligned}$$

and again multiplying the above equation by $u_i^{m,n}(t)$ on both sides and taking sum over $i = 1, 2, \dots, m$, we obtain by using (2.27) that

$$\begin{aligned}
& -\rho_0 \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \Delta \mathbf{u}_{m,n} \right) - \mu \int_{\Omega} \nabla \mathbf{u}_{m,n} \cdot \nabla (\Delta \mathbf{u}_{m,n}) d\mathbf{x} - b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n}) \\
& = - (b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \Delta \mathbf{u}_{m,n}) \\
& \quad - (\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \Delta \mathbf{u}_{m,n}). \tag{2.51}
\end{aligned}$$

According to equation (2.12), we have

$$\left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \Delta \mathbf{u}_{m,n} \right) = - \left(\nabla \frac{\partial \mathbf{u}_{m,n}}{\partial t}, \nabla \mathbf{u}_{m,n} \right),$$

further simplification yields

$$\left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \Delta \mathbf{u}_{m,n} \right) = -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{u}_{m,n}|_2^2 \, d\mathbf{x}.$$

Therefore we can write

$$\left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \Delta \mathbf{u}_{m,n} \right) = -\frac{1}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2. \quad (2.52)$$

Consider the following integral

$$\int_{\Omega} \operatorname{div} (\nabla \mathbf{u}_{m,n} \cdot \Delta \mathbf{u}_{m,n}) \, d\mathbf{x} = \int_{\Omega} (\nabla \mathbf{u}_{m,n} \cdot \nabla \Delta \mathbf{u}_{m,n} + |\Delta \mathbf{u}_{m,n}|_2^2) \, d\mathbf{x},$$

and applying divergence theorem on the left-hand-side of the above equation, we get

$$\int_{\Gamma} \nabla \mathbf{u}_{m,n} \cdot \Delta \mathbf{u}_{m,n} \cdot \mathbf{n} \, d\mathbf{x} = \int_{\Omega} \nabla \mathbf{u}_{m,n} \cdot \nabla \Delta \mathbf{u}_{m,n} \, d\mathbf{x} + \int_{\Omega} |\Delta \mathbf{u}_{m,n}|_2^2 \, d\mathbf{x}.$$

As $\mathbf{u}^{m,n} = 0$ on the boundary Γ and the relations (2.12) and (2.18) reduce the above inequality as

$$\int_{\Omega} \nabla \mathbf{u}_{m,n} \cdot \nabla \Delta \mathbf{u}_{m,n} \, d\mathbf{x} = -|\Delta \mathbf{u}_{m,n}|^2. \quad (2.53)$$

Using equations (2.52)-(2.53) in the equation (2.51), we obtain

$$\begin{aligned} \rho_0 \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 + 2\mu |\Delta \mathbf{u}_{m,n}|^2 &= 2b_u (\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n}) \\ &\quad - 2(b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \Delta \mathbf{u}_{m,n}) \\ &\quad - 2(\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \Delta \mathbf{u}_{m,n}), \end{aligned}$$

or we can write

$$\begin{aligned} \rho_0 \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 + 2\mu |\Delta \mathbf{u}_{m,n}|^2 &\leq |2b_u (\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \\ &\quad + |2(b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \Delta \mathbf{u}_{m,n})| \\ &\quad + |2(\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \Delta \mathbf{u}_{m,n})|. \end{aligned}$$

Making use of hypothesis (H1) – (H5), the above inequality takes the form

$$\begin{aligned} \rho_0 \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 + 2\mu |\Delta \mathbf{u}_{m,n}|^2 &\leq |2b_u (\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \\ &\quad + 2b_1 c_1 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\Delta \mathbf{u}_{m,n}|_2 \, d\mathbf{x} \\ &\quad + 2a_3 \int_{\Omega} |\Delta \mathbf{u}_{m,n}|_2 \, d\mathbf{x}, \end{aligned}$$

and by employing Hölder's inequality, the above inequality becomes

$$\begin{aligned} \rho_0 \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 + 2\mu |\Delta \mathbf{u}_{m,n}|^2 &\leq |2b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \\ &\quad + 2b_1 c_1 |\mathbf{u}_{m,n}| |\Delta \mathbf{u}_{m,n}| \\ &\quad + 2a_3 c_2 |\Delta \mathbf{u}_{m,n}|. \end{aligned}$$

Further by Young's inequality

$$\begin{aligned} \rho_0 \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 + 2\mu |\Delta \mathbf{u}_{m,n}|^2 &\leq |2b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \\ &\quad + \frac{3b_1^2 c_1^2}{\mu} |\mathbf{u}_{m,n}|^2 + \frac{\mu}{3} |\Delta \mathbf{u}_{m,n}|^2 \\ &\quad + \frac{3a_3^2 c_2^2}{\mu} + \frac{\mu}{3} |\Delta \mathbf{u}_{m,n}|^2. \end{aligned} \quad (2.54)$$

Consider now

$$|2b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \leq 2 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\nabla \mathbf{u}_{m,n}|_2 |\Delta \mathbf{u}_{m,n}|_2 dx.$$

Using [55], we have

$$\begin{aligned} |2b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \\ \leq 2c_3 |\mathbf{u}_{m,n}|^{1/2} \|\mathbf{u}_{m,n}\|^{1/2} \|\mathbf{u}_{m,n}\|^{1/2} |\Delta \mathbf{u}_{m,n}|^{1/2} |\Delta \mathbf{u}_{m,n}|, \end{aligned}$$

and simplifying above inequality, we get

$$|2b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \leq 2c_3 |\mathbf{u}_{m,n}|^{1/2} \|\mathbf{u}_{m,n}\| |\Delta \mathbf{u}_{m,n}|^{3/2}.$$

By applying Young's inequality with $p = 4/3$ and $q = 4$, the above inequality takes the following form

$$|2b_u(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \Delta \mathbf{u}_{m,n})| \leq \frac{\mu}{3} |\Delta \mathbf{u}_{m,n}|^2 + \frac{729c_3^4}{16\mu^3} |\mathbf{u}_{m,n}|^2 \|\mathbf{u}_{m,n}\|^4. \quad (2.55)$$

According to the inequality (2.55), the inequality(2.54) becomes

$$\begin{aligned} \rho_0 \frac{d}{dt} \|\mathbf{u}_{m,n}\|^2 + \mu |\Delta \mathbf{u}_{m,n}|^2 &\leq \frac{729c_3^4}{16\mu^3} |\mathbf{u}_{m,n}|^2 \|\mathbf{u}_{m,n}\|^4 \\ &\quad + \frac{3b_1^2 c_1^2}{\mu} |\mathbf{u}_{m,n}|^2 + \frac{3a_3^2 c_2^2}{\mu}, \end{aligned}$$

and with the help of relation (2.39), we arrive

$$\begin{aligned} \rho_0 \frac{d}{dt} \|\mathbf{u}_{m,n}\|^2 + \mu |\Delta \mathbf{u}_{m,n}|^2 &\leq \frac{729c_3^4 c_4}{16\mu^3} \|\mathbf{u}_{m,n}\|^4 \\ &\quad + \frac{3b_1^2 c_1^2 c_5}{\mu} + \frac{3a_3^2 c_2^2}{\mu}. \end{aligned}$$

Let $c_6 = \frac{3b_1^2 c_1^2 c_5}{\mu} + \frac{3a_3^2 c_2^2}{\mu}$ and $c_7 = \frac{729c_3^4 c_4}{16\mu^3}$, we have

$$\rho_0 \frac{d}{dt} \|\mathbf{u}_{m,n}\|^2 + \mu |\Delta \mathbf{u}_{m,n}|^2 \leq c_6 + c_7 \|\mathbf{u}_{m,n}\|^4. \quad (2.56)$$

From the above inequality, we can deduce that

$$\rho_0 \frac{d}{dt} \|\mathbf{u}_{m,n}\|^2 \leq c_6 + c_7 \|\mathbf{u}_{m,n}\|^2 \|\mathbf{u}_{m,n}\|^2,$$

or

$$\frac{d}{dt} \|\mathbf{u}_{m,n}\|^2 \leq c_8 + c_9 \|\mathbf{u}_{m,n}\|^2 \|\mathbf{u}_{m,n}\|^2.$$

Using Gronwall's lemma, we arrive at

$$\begin{aligned} \|\mathbf{u}_{m,n}(t)\|^2 &\leq \|\mathbf{u}_0^{m,n}\|^2 \exp \left(c_9 \int_0^t \|\mathbf{u}_{m,n}(s)\|^2 ds \right) \\ &\quad + c_8 \int_0^t \exp \left(c_9 \int_s^t \|\mathbf{u}_{m,n}(\tau)\|^2 d\tau \right) ds, \end{aligned}$$

and with the help of relations (2.19), (2.39) and as $\mathbf{u}_0 \in (H_0^1(\Omega))^2$, therefore we have

$$\|\mathbf{u}_{m,n}(t)\|^2 \leq c_{10}, \quad \forall t \in (0, T_f).$$

Thus we can conclude that

$$\|\mathbf{u}_{m,n}\|_{L^\infty(0, T_f; V)} \leq c_{10}. \quad (2.57)$$

This shows that $\mathbf{u}_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; V)$.

By integrating the inequality (2.56) over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\rho_0 \int_0^t \frac{d}{ds} \|\mathbf{u}_{m,n}\|^2 ds + \mu \int_0^t |\Delta \mathbf{u}_{m,n}|^2 ds \leq c_6 t + c_7 \int_0^t \|\mathbf{u}_{m,n}\|^4 ds,$$

or

$$\rho_0 \|\mathbf{u}_{m,n}(t)\|^2 + \mu \int_0^t |\Delta \mathbf{u}_{m,n}|^2 ds \leq c_6 t + c_7 \int_0^t \|\mathbf{u}_{m,n}\|^4 ds + \rho_0 \|\mathbf{u}_{m,n}(0)\|^2.$$

From the above inequality, we can deduce that

$$\mu \int_0^t |\Delta \mathbf{u}_{m,n}|^2 ds \leq c_6 t + c_7 \int_0^t \|\mathbf{u}_{m,n}\|^4 ds + \rho_0 \|\mathbf{u}_{m,n}(0)\|^2, \quad \forall t \in (0, T_f)$$

Making use of results (2.19), (2.57) and as $\mathbf{u}_0 \in (H_0^1(\Omega))^2$, we obtain

$$\int_0^{T_f} |\Delta \mathbf{u}_{m,n}|^2 dt \leq c_{11}.$$

This implies that

$$\|\Delta \mathbf{u}_{m,n}\|_{L^2(0,T_f;H)} \leq c_{11}, \quad (2.58)$$

which shows that $\Delta \mathbf{u}_{m,n}$ is uniformly bounded in $L^2(0, T_f; H)$. Therefore using the relation (2.39) and the elliptic estimate, we can deduce that

$$\|\mathbf{u}_{m,n}\|_{L^2(0,T_f;H^2(\Omega))} \leq c_{12}. \quad (2.59)$$

Now multiplying equation (2.22) by $\partial u_i^{m,n}(t)/\partial t$ and then taking sum over $i = 1, 2, \dots, m$, we have

$$\begin{aligned} & \rho_0 \left(\frac{\partial \mathbf{u}_{m,n}}{\partial t}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) + a_u \left(\mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) + b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \\ &= \left(\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) + \left(b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right), \end{aligned}$$

and by employing Green's formula, we arrive at

$$\begin{aligned} & \rho_0 \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 + b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \\ &= \left(b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \\ &\quad + \left(\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right), \end{aligned}$$

or

$$\begin{aligned} & \rho_0 \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 \leq \left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \\ &\quad + \left| \left(b(\psi_{m,n})((\mathbf{u}_{m,n} \times \mathbf{B}) \times \mathbf{B}), \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \\ &\quad + \left| \left(\mathbf{A}_1(\psi_{m,n}, c_{m,n}), \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right|. \end{aligned}$$

Using hypothesis (H1) – (H5), the above inequality takes the form

$$\begin{aligned} & \rho_0 \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 \leq \left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \\ &\quad + b_1 c_{13} \int_{\Omega} |\mathbf{u}_{m,n}|_2 \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|_2 d\mathbf{x} + a_3 \int_{\Omega} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|_2 d\mathbf{x}, \end{aligned}$$

with the help of Hölder's inequality, the above inequality becomes

$$\begin{aligned} & \rho_0 \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 \leq \left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \\ &\quad + b_1 c_{13} |\mathbf{u}_{m,n}| \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right| + a_3 c_{14} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|, \end{aligned}$$

and further by Young's inequality, we arrive at

$$\begin{aligned} \rho_0 \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 &\leq \left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \\ &\quad + \frac{3b_1^2 c_{13}^2}{2\rho_0} |\mathbf{u}_{m,n}|^2 + \frac{\rho_0}{6} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 \\ &\quad + \frac{3a_3^2 c_{14}^2}{2\rho_0} + \frac{\rho_0}{6} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2. \end{aligned} \quad (2.60)$$

As we know that

$$b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) = \rho_0 \int_{\Omega} \mathbf{u}_{m,n} \cdot \nabla \mathbf{u}_{m,n} \frac{\partial \mathbf{u}_{m,n}}{\partial t} d\mathbf{x},$$

and employing Hölder's inequality, we can write

$$\left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \leq \rho_0 \|\mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla \mathbf{u}_{m,n}\|_{L^4(\Omega)} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|,$$

and making use of Sobolev injections, we obtain

$$\left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \leq \rho_0 c_{15} \|\mathbf{u}_{m,n}\| \|\mathbf{u}_{m,n}\|_{H^2(\Omega)} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|.$$

According to the relation (2.57), we have

$$\left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \leq \rho_0 c_{16} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|,$$

and with the help of Young's inequality, we arrive at

$$\left| b_u \left(\mathbf{u}_{m,n}, \mathbf{u}_{m,n}, \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right) \right| \leq \frac{\rho_0}{6} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{3\rho_0 c_{16}^2}{2} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2. \quad (2.61)$$

By the relation (2.61), the inequality (2.60) takes the form

$$\begin{aligned} \frac{\rho_0}{2} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 &\leq \frac{3b_1^2 c_{13}^2}{2\rho_0} |\mathbf{u}_{m,n}|^2 + \frac{3a_3^2 c_{14}^2}{2\rho_0} \\ &\quad + \frac{3\rho_0 c_{16}^2}{2} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2, \end{aligned}$$

and the relation (2.57) reduces the above inequality to the following form

$$\begin{aligned} \frac{\rho_0}{2} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 &\leq \frac{3b_1^2 c_{13}^2 c_{17}}{2\rho_0} + \frac{3a_3^2 c_{14}^2}{2\rho_0} \\ &\quad + \frac{3\rho_0 c_{16}^2}{2} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2. \end{aligned}$$

Let $c_{18} = \max \left(\frac{3b_1^2 c_{13}^2 c_{17}}{2\rho_0} + \frac{3a_3^2 c_{14}^2}{2\rho_0}, \frac{3\rho_0 c_{16}^2}{2} \right)$, then

$$\frac{\rho_0}{2} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}_{m,n}|^2 \leq c_{18} \left(1 + \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 \right).$$

Integrating with respect to time over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\frac{\rho_0}{2} \int_0^t \left| \frac{\partial \mathbf{u}_{m,n}}{\partial s} \right|^2 ds + \frac{\mu}{2} \int_0^t \frac{d}{ds} |\nabla \mathbf{u}_{m,n}|^2 ds \leq c_{18} \int_0^t \left(1 + \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 \right) ds,$$

or

$$\begin{aligned} \frac{\rho_0}{2} \int_0^t \left| \frac{\partial \mathbf{u}_{m,n}}{\partial s} \right|^2 ds + \frac{\mu}{2} |\nabla \mathbf{u}_{m,n}(t)|^2 &\leq c_{18} \int_0^t \left(1 + \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 \right) ds \\ &\quad + \frac{\mu}{2} |\nabla \mathbf{u}_{m,n}(0)|^2, \quad \forall t \in (0, T_f). \end{aligned}$$

From above inequality we can deduce that

$$\begin{aligned} \frac{\rho_0}{2} \int_0^{T_f} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial s} \right|^2 ds &\leq c_{18} \int_0^{T_f} \left(1 + \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 \right) ds \\ &\quad + \frac{\mu}{2} |\nabla \mathbf{u}_{m,n}(0)|^2. \end{aligned}$$

According to the relations (2.19) and (2.59) and as $\mathbf{u}_0 \in (H_0^1(\Omega))^2$, we finally get

$$\frac{\rho_0}{2} \int_0^{T_f} \left| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right|^2 dt \leq c_{19},$$

this implies that

$$\left\| \frac{\partial \mathbf{u}_{m,n}}{\partial t} \right\|_{L^2(0, T_f; H)} \leq c_{20}. \quad (2.62)$$

This shows that $\frac{\partial \mathbf{u}_{m,n}}{\partial t}$ is uniformly bounded in $L^2(0, T_f; H)$.

Let us define the following space

$$\mathcal{W}_2 = \left\{ v \in L^2(0, T_f; H^2(\Omega)), \frac{\partial v}{\partial t} \in L^2(0, T_f; L^2(\Omega)) \right\}. \quad (2.63)$$

We deduce from the relations (2.59) and (2.62) that when $\mathbf{u}_0 \in (H_0^1(\Omega))^2$, then the sequence $\mathbf{u}_{m,n}$ is uniformly bounded in $(\mathcal{W}_2)^2$. Since the embedding of $H^2(\Omega)$ into $H^1(\Omega)$ is compact, we conclude $(\mathcal{W}_2)^2$ is compactly embedded into $L^2(0, T_f; H^1(\Omega))$, (see e.g. [55]). Therefore there exists a subsequence of the sequence $\mathbf{u}_{m,n}$, also denoted by $\mathbf{u}_{m,n}$, such that as $m \rightarrow \infty$ we have

$$\begin{aligned} \mathbf{u}_{m,n} &\rightarrow \mathbf{u}, \text{ strongly in } L^2(0, T_f; (H^1(\Omega))^2), \\ \mathbf{u}_{m,n} &\rightharpoonup \mathbf{u}, \text{ weakly in } L^2(0, T_f; (H^2(\Omega))^2), \\ \frac{\partial \mathbf{u}_{m,n}}{\partial t} &\rightharpoonup \frac{\partial \mathbf{u}}{\partial t}, \text{ weakly in } L^2(0, T_f; (L^2(\Omega))^2). \end{aligned}$$

Therefore we conclude that

$$\mathbf{u} \in L^2(0, T_f; (H^2(\Omega))^2) \cap H^1(0, T_f; (L^2(\Omega))^2).$$

Now Multiplying equation (2.23) by μ_k on both sides and using equation (2.13), we get

$$\begin{aligned} - \left(\frac{\partial \psi_{m,n}}{\partial t}, \Delta e_k \right) - \epsilon_1 \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta e_k d\mathbf{x} - b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta e_k) \\ = \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \Delta e_k d\mathbf{x}. \end{aligned} \quad (2.64)$$

Again multiplying above equation by $\psi_k^{m,n}(t)$ on both sides and then taking sum over k , where $k = 1, 2, \dots, n$, we obtain by using relation (2.27)

$$\begin{aligned} - \left(\frac{\partial \psi_{m,n}}{\partial t}, \Delta \psi_{m,n} \right) - \epsilon_1 \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_{m,n} d\mathbf{x} - b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_{m,n}) \\ = \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \Delta \psi_{m,n} d\mathbf{x}. \end{aligned} \quad (2.65)$$

Consider the following integral

$$\int_{\Omega} \operatorname{div} \left(\frac{\partial \psi_{m,n}}{\partial t} \nabla \psi_{m,n} \right) d\mathbf{x} = \int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_{m,n} d\mathbf{x} + \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_{m,n} d\mathbf{x},$$

and by employing divergence theorem on the left-hand-side of the above equation, we have

$$\int_{\Gamma} \frac{\partial \psi_{m,n}}{\partial t} \nabla \psi_{m,n} \cdot \mathbf{n} d\Gamma = \int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_{m,n} d\mathbf{x} + \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_{m,n} d\mathbf{x}.$$

Since $\nabla \psi_{m,n} \cdot \mathbf{n} = 0$, therefore we have

$$\int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_{m,n} d\mathbf{x} = - \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_{m,n} d\mathbf{x},$$

or

$$\int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_{m,n} d\mathbf{x} = - \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi_{m,n}|^2 d\mathbf{x}.$$

The above equation can further be written as

$$\int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_{m,n} d\mathbf{x} = - \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi_{m,n}|^2 d\mathbf{x}. \quad (2.66)$$

Consider again the following integral

$$\int_{\Omega} \operatorname{div} (\nabla \psi_{m,n} \Delta \psi_{m,n}) d\mathbf{x} = \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_{m,n} d\mathbf{x} + \int_{\Omega} |\Delta \psi_{m,n}|^2 d\mathbf{x},$$

and by divergence theorem, we arrive at

$$\int_{\Gamma} \Delta \psi_{m,n} \nabla \psi_{m,n} \cdot \mathbf{n} \, d\Gamma = \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_{m,n} \, d\mathbf{x} + \int_{\Omega} |\Delta \psi_{m,n}|^2 \, d\mathbf{x}.$$

Since $\nabla \psi_{m,n} \cdot \mathbf{n} = 0$, therefore we have

$$\int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_{m,n} \, d\mathbf{x} = - \int_{\Omega} |\Delta \psi_{m,n}|^2 \, d\mathbf{x},$$

the above equation can be written as

$$\int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_{m,n} \, d\mathbf{x} = - |\Delta \psi_{m,n}|^2. \quad (2.67)$$

Making use of equations (2.66)-(2.67) in equation (2.65), we arrive at

$$\begin{aligned} \frac{d}{dt} |\nabla \psi_{m,n}|^2 + 2\epsilon_1 |\Delta \psi_{m,n}|^2 &= 2 \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \Delta \psi_{m,n} \, d\mathbf{x} \\ &\quad + 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_{m,n}), \end{aligned}$$

or

$$\begin{aligned} \frac{d}{dt} |\nabla \psi_{m,n}|^2 + 2\epsilon_1 |\Delta \psi_{m,n}|^2 &\leq 2 \int_{\Omega} |A_2(\psi_{m,n}, c_{m,n})|_2 |\Delta \psi_{m,n}|_2 \, d\mathbf{x} \\ &\quad + |2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_{m,n})|. \end{aligned} \quad (2.68)$$

Consider the term

$$|2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_{m,n})| \leq 2 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\nabla \psi_{m,n}|_2 |\Delta \psi_{m,n}| \, d\mathbf{x},$$

by employing Hölder's inequality, we have

$$|2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_{m,n})| \leq 2 \|\mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} |\Delta \psi_{m,n}|.$$

According to the Lemma 1, we can write

$$\begin{aligned} |2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_{m,n})| &\leq c_{21} \|\mathbf{u}_{m,n}\| |\nabla \psi_{m,n}|^{1/2} \|\nabla \psi_{m,n}\|^{1/2} |\Delta \psi_{m,n}|, \end{aligned}$$

and with the help of Young's inequality, we obtain

$$\begin{aligned} |2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_{m,n})| &\leq \frac{c_{21}^2}{\epsilon_1} \|\mathbf{u}_{m,n}\|^2 \|\nabla \psi_{m,n}\| \|\nabla \psi_{m,n}\| \\ &\quad + \frac{\epsilon_1}{4} |\Delta \psi_{m,n}|^2, \end{aligned}$$

The relation (2.57) reduces the above inequality to the following form

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_{m,n})| \leq c_{22} |\nabla\psi_{m,n}| \|\nabla\psi_{m,n}\| + \frac{\epsilon_1}{4} |\Delta\psi_{m,n}|^2.$$

By the definition of Sobolev norm, the above inequality can be written as

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_{m,n})| \leq c_{23} |\nabla\psi_{m,n}| \|\psi_{m,n}\|_{H^2(\Omega)} + \frac{\epsilon_1}{4} |\Delta\psi_{m,n}|^2,$$

and using elliptic estimate (see Lemma 1), we have

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_{m,n})| \leq c_{24} |\nabla\psi_{m,n}| (|\psi_{m,n}| + |\Delta\psi_{m,n}|) + \frac{\epsilon_1}{4} |\Delta\psi_{m,n}|^2,$$

or

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_{m,n})| \leq c_{24} |\nabla\psi_{m,n}| |\psi_{m,n}| + c_{24} |\nabla\psi_{m,n}| |\Delta\psi_{m,n}| + \frac{\epsilon_1}{4} |\Delta\psi_{m,n}|^2.$$

By the Young's inequality, we arrive at

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_{m,n})| \leq \frac{c_{24}^2}{2} |\nabla\psi_{m,n}|^2 + \frac{1}{2} |\psi_{m,n}|^2 + \frac{c_{24}^2}{\epsilon_1} |\nabla\psi_{m,n}|^2 + \frac{\epsilon_1}{4} |\Delta\psi_{m,n}|^2 + \frac{\epsilon_1}{4} |\Delta\psi_{m,n}|^2,$$

and simplification of the above inequality yields

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_{m,n})| \leq c_{25} |\nabla\psi_{m,n}|^2 + \frac{1}{2} |\psi_{m,n}|^2 + \frac{\epsilon_1}{2} |\Delta\psi_{m,n}|^2. \quad (2.69)$$

Using hypothesis (H1) – (H5) and (2.69), the inequality (2.68) takes the form

$$\begin{aligned} \frac{d}{dt} |\nabla\psi_{m,n}|^2 + 2\epsilon_1 |\Delta\psi_{m,n}|^2 &\leq 2a_2 \int_{\Omega} |\Delta\psi_{m,n}| d\mathbf{x} + c_{25} |\nabla\psi_{m,n}|^2 + \frac{1}{2} |\psi_{m,n}|^2 \\ &\quad + \frac{\epsilon_1}{2} |\Delta\psi_{m,n}|^2, \end{aligned}$$

and then (by using Young's inequality)

$$\begin{aligned} \frac{d}{dt} |\nabla\psi_{m,n}|^2 + 2\epsilon_1 |\Delta\psi_{m,n}|^2 &\leq c_{26} + \frac{\epsilon_1}{2} |\Delta\psi_{m,n}|^2 + c_{25} |\nabla\psi_{m,n}|^2 \\ &\quad + \frac{1}{2} |\psi_{m,n}|^2 + \frac{\epsilon_1}{2} |\Delta\psi_{m,n}|^2. \end{aligned}$$

According to the relation (2.36), the above inequality takes the form

$$\frac{d}{dt} |\nabla\psi_{m,n}|^2 + \epsilon_1 |\Delta\psi_{m,n}|^2 \leq c_{27} + c_{25} |\nabla\psi_{m,n}|^2. \quad (2.70)$$

From the above inequality we can deduce that

$$\frac{d}{dt} |\nabla \psi_{m,n}|^2 \leq c_{27} + c_{25} |\nabla \psi_{m,n}|^2.$$

Making use of Gronwall's lemma, we arrive at

$$|\nabla \psi_{m,n}(t)|^2 \leq |\nabla \psi_{m,n}(0)|^2 e^{c_{25}t} + c_{28}, \quad \forall t \in [0, T_f].$$

Since $\psi_0 \in H^1(\Omega)$, therefore using the relation (2.19), we finally obtain

$$|\nabla \psi_{m,n}(t)|^2 \leq c_{29}, \quad \forall t \in (0, T_f).$$

Thus we have

$$\|\nabla \psi_{m,n}\|_{L^\infty(0, T_f; L^2(\Omega))} \leq c_{29}.$$

This shows that $\nabla \psi_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; L^2(\Omega))$. Therefore with the help of above inequality and the result (2.39) we can easily have

$$\|\psi_{m,n}\|_{L^\infty(0, T_f; H^1(\Omega))} \leq c_{30}. \quad (2.71)$$

This shows that $\psi_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; H^1(\Omega))$.

Now integrating the relation (2.70) with respect to time over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\int_0^t \frac{d}{ds} |\nabla \psi_{m,n}|^2 ds + \epsilon_1 \int_0^t |\Delta \psi_{m,n}|^2 ds \leq c_{31}t + c_{25} \int_0^t |\nabla \psi_{m,n}|^2 ds,$$

or

$$|\nabla \psi_{m,n}(t)|^2 + \epsilon_1 \int_0^t |\Delta \psi_{m,n}|^2 ds \leq c_{31} + c_{25} \int_0^t |\nabla \psi_{m,n}|^2 ds + |\nabla \psi_{m,n}(0)|^2,$$

from above inequality, we can write

$$\epsilon_1 \int_0^t |\Delta \psi_{m,n}|^2 ds \leq c_{31}t + c_{25} \int_0^t |\nabla \psi_{m,n}|^2 ds + |\nabla \psi_{m,n}(0)|^2, \quad \forall t \in (0, T_f).$$

According to the relations (2.20), (2.71) and as $\psi_0 \in H^1(\Omega)$, the above inequality reduces to

$$\epsilon_1 \int_0^t |\Delta \psi_{m,n}|^2 ds \leq c_{32}, \quad \forall t \in (0, T_f)$$

or

$$\int_0^{T_f} |\Delta \psi_{m,n}|^2 ds \leq c_{33}.$$

Thus we have the following result

$$\|\Delta\psi_{m,n}\|_{L^2(0,T_f;L^2(\Omega))} \leq c_{33}.$$

Therefore with the help of the above inequality, elliptic estimate and the relation (2.39), we can easily prove that

$$\|\psi_{m,n}\|_{L^2(0,T_f;H^2(\Omega))} \leq c_{34}. \quad (2.72)$$

This implies that $\psi_{m,n}$ is uniformly bounded in $L^2(0, T_f; H^2(\Omega))$.

Multiplying equation (2.23) by $\partial\psi_k^{m,n}/\partial t$ on both sides and then taking sum over k , where $k = 1, 2, \dots, n$, we obtain by using relation (2.27)

$$\begin{aligned} \left(\frac{\partial\psi_{m,n}}{\partial t}, \frac{\partial\psi_{m,n}}{\partial t} \right) + \epsilon_1 \int_{\Omega} \nabla\psi_{m,n} \cdot \nabla \frac{\partial\psi_{m,n}}{\partial t} d\mathbf{x} + b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial\psi_{m,n}}{\partial t}) \\ = \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \frac{\partial\psi_{m,n}}{\partial t} d\mathbf{x}, \end{aligned}$$

and the above equation can further be written as

$$\begin{aligned} \left| \frac{\partial\psi_{m,n}}{\partial t} \right|^2 + \frac{\epsilon_1}{2} \frac{d}{dt} |\nabla\psi_{m,n}|^2 = \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \frac{\partial\psi_{m,n}}{\partial t} d\mathbf{x} \\ - b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial\psi_{m,n}}{\partial t}), \end{aligned}$$

or

$$\begin{aligned} \left| \frac{\partial\psi_{m,n}}{\partial t} \right|^2 + \frac{\epsilon_1}{2} \frac{d}{dt} \int_{\Omega} |\nabla\psi_{m,n}|^2 d\mathbf{x} \leq \int_{\Omega} |A_2(\psi_{m,n}, c_{m,n})| \left| \frac{\partial\psi_{m,n}}{\partial t} \right| d\mathbf{x} \\ + \left| b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial\psi_{m,n}}{\partial t}) \right|. \end{aligned}$$

Using hypothesis (H1) – (H5), the above inequality takes the form

$$\left| \frac{\partial\psi_{m,n}}{\partial t} \right|^2 + \frac{\epsilon_1}{2} \frac{d}{dt} |\nabla\psi_{m,n}|^2 \leq a_2 \int_{\Omega} \left| \frac{\partial\psi_{m,n}}{\partial t} \right| d\mathbf{x} + \left| b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial\psi_{m,n}}{\partial t}) \right|,$$

and then

$$\begin{aligned} 2 \left| \frac{\partial\psi_{m,n}}{\partial t} \right|^2 + \epsilon_1 \frac{d}{dt} |\nabla\psi_{m,n}|^2 \leq \left| 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial\psi_{m,n}}{\partial t}) \right| \\ + c_{35} \left| \frac{\partial\psi_{m,n}}{\partial t} \right|. \end{aligned} \quad (2.73)$$

Consider now

$$\left| 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial\psi_{m,n}}{\partial t}) \right| \leq 2 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\nabla\psi_{m,n}|_2 \left| \frac{\partial\psi_{m,n}}{\partial t} \right| d\mathbf{x},$$

and employing the Hölder's inequality, we have

$$\left| 2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_{m,n}}{\partial t}) \right| \leq 2 \|\mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} \left| \frac{\partial \psi_{m,n}}{\partial t} \right|.$$

According to the Lemma 1, we can write

$$\left| 2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_{m,n}}{\partial t}) \right| \leq c_{36} \|\mathbf{u}_{m,n}\| \|\psi_{m,n}\|_{H^2(\Omega)} \left| \frac{\partial \psi_{m,n}}{\partial t} \right|,$$

and with the help of the relation (2.57) and Young's inequality, we arrive at

$$\left| 2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_{m,n}}{\partial t}) \right| \leq c_{37} \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{1}{2} \left| \frac{\partial \psi_{m,n}}{\partial t} \right|^2. \quad (2.74)$$

Using the inequality (2.74) into the inequality (2.73), we arrive at

$$\begin{aligned} 2 \left| \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \epsilon_1 \frac{d}{dt} |\nabla \psi_{m,n}|^2 &\leq c_{35} \left| \frac{\partial \psi_{m,n}}{\partial t} \right| + c_{37} \|\psi_{m,n}\|_{H^2(\Omega)}^2 \\ &\quad + \frac{1}{2} \left| \frac{\partial \psi_{m,n}}{\partial t} \right|^2. \end{aligned}$$

Applying Young's inequality on the first term of the right-hand-side of the above inequality and then simplifying we get

$$\left| \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \epsilon_1 \frac{d}{dt} |\nabla \psi_{m,n}|^2 \leq c_{38} + c_{37} \|\psi_{m,n}\|_{H^2(\Omega)}^2,$$

and integrating over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\int_0^t \left| \frac{\partial \psi_{m,n}}{\partial \tau} \right|^2 d\tau + \epsilon_1 \int_0^t \frac{d}{ds} |\nabla \psi_{m,n}|^2 ds \leq c_{39} + c_{37} \int_0^t \|\psi_{m,n}\|_{H^2(\Omega)}^2 ds.$$

With the help of the relation (2.72), we conclude that

$$\int_0^t \left| \frac{\partial \psi_{m,n}}{\partial \tau} \right|^2 d\tau + \epsilon_1 \int_0^t \frac{d}{ds} |\nabla \psi_{m,n}|^2 ds \leq c_{40},$$

or

$$\int_0^t \left| \frac{\partial \psi_{m,n}}{\partial \tau} \right|^2 d\tau + \epsilon_1 |\nabla \psi_{m,n}(t)|^2 \leq c_{40} + \epsilon_1 |\nabla \psi_0^{m,n}|^2.$$

Since $\psi_0 \in H^1(\Omega)$ and using the result (2.19), we have

$$\int_0^{T_f} \left| \frac{\partial \psi_{m,n}}{\partial \tau} \right|^2 d\tau \leq c_{41},$$

or

$$\left\| \frac{\partial \psi_{m,n}}{\partial t} \right\|_{L^2(0, T_f; L^2(\Omega))} \leq c_{41}. \quad (2.75)$$

This shows that $\frac{\partial \psi_{m,n}}{\partial t}$ is uniformly bounded in $L^2(0, T_f; L^2(\Omega))$.

From equations (2.72) and (2.75), we can deduce that if $\psi_0 \in H^1(\Omega)$ then $\psi_{m,n}$ is uniformly bounded in \mathcal{W}_2 (defined in (2.63)). As we know that the embedding of $H^2(\Omega)$ in $H^1(\Omega)$ is compact, therefore we conclude that \mathcal{W}_2 is compactly embedded into $L^2(0, T_f; H^1(\Omega))$, (see e.g., [55]). Therefore we can extract from $\psi_{m,n}$, a subsequence also denoted by $\psi_{m,n}$, such that, as $m, n \rightarrow \infty$ we have

$$\begin{aligned} \psi_{m,n} &\rightarrow \psi \text{ strongly in } L^2(0, T_f; H^1(\Omega)), \\ \psi_{m,n} &\rightharpoonup \psi \text{ weakly in } L^2(0, T_f; H^2(\Omega)), \\ \frac{\partial \psi_{m,n}}{\partial t} &\rightharpoonup \frac{\partial \psi}{\partial t} \text{ weakly in } L^2(0, T_f; L^2(\Omega)), \end{aligned}$$

and therefore we conclude that

$$\psi \in L^2(0, T_f; H^2(\Omega)) \cap H^1(0, T_f; L^2(\Omega)).$$

We can then pass easily limit, $m, n \rightarrow \infty$ in the problem (2.22)-(2.25) and verify that (\mathbf{u}, ψ, c) satisfy the problem (2.11). \square

Theorem 3 Let $(\mathbf{u}_0, \psi_0, c_0) \in (H_0^1(\Omega))^2 \times H_0^2(\Omega) \times H^1(\Omega)$, and the assumptions (H1) – (H5) are satisfied. Then there exist a triplet of functions (\mathbf{u}, ψ, c) satisfying

$$\begin{aligned} \mathbf{u} &\in L^2\left(0, T_f; (H^2(\Omega))^2\right) \cap H^1\left(0, T_f; (L^2(\Omega))^2\right), \\ \psi &\in L^2(0, T_f; H^3(\Omega)) \cap H^1(0, T_f; H^1(\Omega)), \\ c &\in L^2(0, T_f; H^2(\Omega)) \cap H^1(0, T_f; L^2(\Omega)), \end{aligned}$$

which is a solution of the problem (2.11).

Proof: Multiplying equation (2.23) by μ_k^2 on both sides and using equation (2.13), we get

$$\begin{aligned} \left(\frac{\partial \psi_{m,n}}{\partial t}, \Delta^2 e_k \right) + \epsilon_1 \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta^2 e_k d\mathbf{x} + b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta^2 e_k) \\ = - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \Delta^2 e_k d\mathbf{x}, \end{aligned} \quad (2.76)$$

again multiplying above equation by $\psi_k^{m,n}$ on both sides and then taking sum over k , where $k = 1, 2, \dots, n$, we obtain by using the relation (2.27)

$$\begin{aligned} \left(\frac{\partial \psi_{m,n}}{\partial t}, \Delta \psi_d^{m,n} \right) + \epsilon_1 \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_d^{m,n} d\mathbf{x} + b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n}) \\ = - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \Delta \psi_d^{m,n} d\mathbf{x} \end{aligned} \quad (2.77)$$

where $\psi_d^{m,n} = \Delta \psi_{m,n}$. Consider the following integral

$$\int_{\Omega} \operatorname{div} \left(\frac{\partial \psi_{m,n}}{\partial t} \nabla \psi_d^{m,n} \right) d\mathbf{x} = \int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_d^{m,n} d\mathbf{x} + \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_d^{m,n} d\mathbf{x},$$

and employing the divergence theorem on the left-hand-side of the above equation, we have

$$\int_{\Gamma} \frac{\partial \psi_{m,n}}{\partial t} \nabla \psi_d^{m,n} \cdot \mathbf{n} d\Gamma = \int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_d^{m,n} d\mathbf{x} + \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_d^{m,n} d\mathbf{x}.$$

According to the relations (2.13) and (2.27), we have $\nabla \psi_d^{m,n} \cdot \mathbf{n} = 0$ on the boundary Γ , therefore we get

$$\int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_d^{m,n} d\mathbf{x} = - \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_d^{m,n} d\mathbf{x}. \quad (2.78)$$

Consider again

$$\begin{aligned} \int_{\Omega} \operatorname{div} \left(\nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \psi_d^{m,n} \right) d\mathbf{x} &= \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_d^{m,n} d\mathbf{x} \\ &\quad + \int_{\Omega} \Delta \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \psi_d^{m,n} d\mathbf{x}, \end{aligned}$$

and applying the divergence theorem, we have

$$\begin{aligned} \int_{\Gamma} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \psi_d^{m,n} \cdot \mathbf{n} d\Gamma &= \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_d^{m,n} d\mathbf{x} \\ &\quad + \int_{\Omega} \Delta \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \psi_d^{m,n} d\mathbf{x}. \end{aligned}$$

Since $\nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \mathbf{n} = 0$ on Γ (according to (2.27)), therefore we have

$$\int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_d^{m,n} d\mathbf{x} = - \int_{\Omega} \Delta \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \psi_d^{m,n} d\mathbf{x},$$

and the above equation can further be written as

$$\int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \cdot \nabla \psi_d^{m,n} d\mathbf{x} = - \frac{1}{2} \frac{d}{dt} |\psi_d^{m,n}|^2. \quad (2.79)$$

From the equations (2.78) and (2.79), we conclude that

$$\int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \Delta \psi_d^{m,n} d\mathbf{x} = \frac{1}{2} \frac{d}{dt} |\psi_d^{m,n}|^2. \quad (2.80)$$

Consider the following integral

$$\int_{\Omega} \operatorname{div} (\nabla \psi_{m,n} \Delta \psi_d^{m,n}) d\mathbf{x} = \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_d^{m,n} d\mathbf{x} + \int_{\Omega} \Delta \psi_{m,n} \Delta \psi_d^{m,n} d\mathbf{x},$$

by employing divergence theorem on the left-hand-side of the above equation, we get

$$\int_{\Gamma} \nabla \psi_{m,n} \Delta \psi_d^{m,n} \cdot \mathbf{n} d\Gamma = \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_d^{m,n} d\mathbf{x} + \int_{\Omega} \Delta \psi_{m,n} \Delta \psi_d^{m,n} d\mathbf{x}.$$

As we know that $\nabla \psi_{m,n} \cdot \mathbf{n} = 0$ on the boundary Γ , thus we have

$$\int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_d^{m,n} d\mathbf{x} = - \int_{\Omega} \Delta \psi_{m,n} \Delta \psi_d^{m,n} d\mathbf{x}. \quad (2.81)$$

Consider again

$$\int_{\Omega} \operatorname{div} (\Delta \psi_{m,n} \nabla \psi_d^{m,n}) d\mathbf{x} = \int_{\Omega} \Delta \psi_{m,n} \Delta \psi_d^{m,n} d\mathbf{x} + \int_{\Omega} \nabla (\Delta \psi_{m,n}) \cdot \nabla \psi_d^{m,n} d\mathbf{x},$$

and using the divergence theorem, we have

$$\int_{\Gamma} \Delta \psi_{m,n} \nabla \psi_d^{m,n} \cdot \mathbf{n} d\Gamma = \int_{\Omega} \Delta \psi_{m,n} \Delta \psi_d^{m,n} d\mathbf{x} + \int_{\Omega} |\nabla \psi_d^{m,n}|_2^2 d\mathbf{x}.$$

Since $\nabla \psi_d^{m,n} \cdot \mathbf{n} = 0$ on Γ , therefore

$$\int_{\Omega} \Delta \psi_{m,n} \Delta \psi_d^{m,n} d\mathbf{x} = - |\nabla \psi_d^{m,n}|^2. \quad (2.82)$$

From equations (2.81) and (2.82), we can deduce that

$$\int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \Delta \psi_d^{m,n} d\mathbf{x} = |\nabla \psi_d^{m,n}|^2. \quad (2.83)$$

Making use of equations (2.80) and (2.83) in equation (2.77), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\psi_d^{m,n}|^2 + \epsilon_1 |\nabla \psi_d^{m,n}|^2 &= - \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \Delta \psi_d^{m,n} d\mathbf{x} \\ &\quad - b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n}), \end{aligned}$$

and by applying the Green's theorem on the first term of right-hand-side of the above equation, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\psi_d^{m,n}|^2 + \epsilon_1 |\nabla \psi_d^{m,n}|^2 &= \int_{\Omega} \nabla A_2(\psi_{m,n}, c_{m,n}) \cdot \nabla \psi_d^{m,n} d\mathbf{x} \\ &\quad - b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n}). \end{aligned}$$

From the above equation, we have

$$\begin{aligned} \frac{d}{dt} |\psi_d^{m,n}|^2 + 2\epsilon_1 |\nabla \psi_d^{m,n}|^2 &\leq 2 \int_{\Omega} |\nabla A_2(\psi_{m,n}, c_{m,n})|_2 |\nabla \psi_d^{m,n}|_2 d\mathbf{x} \\ &\quad + |2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n})|. \end{aligned} \quad (2.84)$$

Consider now

$$\begin{aligned} \int_{\Omega} \operatorname{div} ((\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \psi_d^{m,n}) d\mathbf{x} &= \int_{\Omega} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \Delta \psi_d^{m,n} d\mathbf{x} \\ &\quad + \int_{\Omega} \nabla (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \psi_d^{m,n} d\mathbf{x}, \end{aligned}$$

and then (using divergence theorem)

$$\begin{aligned} \int_{\Gamma} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \psi_d^{m,n} \cdot \mathbf{n} d\Gamma &= \int_{\Omega} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \Delta \psi_d^{m,n} d\mathbf{x} \\ &\quad + \int_{\Omega} \nabla (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \psi_d^{m,n} d\mathbf{x}. \end{aligned}$$

Since $\nabla \psi_{m,n} \cdot \mathbf{n} = 0$ on the boundary Γ , therefore we obtain

$$\int_{\Omega} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \Delta \psi_d^{m,n} d\mathbf{x} = - \int_{\Omega} \nabla (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \psi_d^{m,n} d\mathbf{x},$$

and we can write the above equation as

$$b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n}) = - \int_{\Omega} (\nabla \mathbf{u}_{m,n} \cdot \nabla \psi_{m,n} + \nabla (\nabla \psi_{m,n}) \cdot \mathbf{u}_{m,n}) \cdot \nabla \psi_d^{m,n} d\mathbf{x},$$

then

$$\begin{aligned} |2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n})| &\leq 2 \int_{\Omega} |\nabla \mathbf{u}_{m,n}|_2 |\nabla \psi_{m,n}|_2 |\nabla \psi_d^{m,n}|_2 d\mathbf{x} \\ &\quad + 2 \int_{\Omega} |\nabla (\nabla \psi_{m,n})|_2 |\mathbf{u}_{m,n}|_2 |\nabla \psi_d^{m,n}|_2 d\mathbf{x}, \end{aligned}$$

and (using Hölder's inequality), we have

$$\begin{aligned} |2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n})| &\leq 2 \|\nabla \mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} |\nabla \psi_d^{m,n}| \\ &\quad + 2 \|\mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla (\nabla \psi_{m,n})\|_{L^4(\Omega)} |\nabla \psi_d^{m,n}|. \end{aligned}$$

Making use of Lemma 1, the above inequality takes the form

$$\begin{aligned} |2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta \psi_d^{m,n})| &\leq c_1 |\nabla \mathbf{u}_{m,n}|^{1/2} \|\nabla \mathbf{u}_{m,n}\|^{1/2} |\nabla \psi_{m,n}|^{1/2} \|\nabla \psi_{m,n}\|^{1/2} |\nabla \psi_d^{m,n}| \\ &\quad + c_2 \|\mathbf{u}_{m,n}\| |\nabla (\nabla \psi_{m,n})|^{1/2} \|\nabla (\nabla \psi_{m,n})\|^{1/2} |\nabla \psi_d^{m,n}|. \end{aligned}$$

According to the relations (2.57), (2.71) and the definition of Sobolev norm, the above inequality takes the form

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_d^{m,n})| \leq c_3 \|\nabla \mathbf{u}_{m,n}\|^{1/2} \|\nabla \psi_{m,n}\|^{1/2} |\nabla \psi_d^{m,n}| \\ + c_4 \|\psi_{m,n}\|_{H^2(\Omega)}^{1/2} \|\nabla \psi_{m,n}\|_{H^2(\Omega)}^{1/2} |\nabla \psi_d^{m,n}|,$$

and then with the help of Young's inequality, we arrive at

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_d^{m,n})| \leq \frac{3c_3^2}{2\epsilon_1} \|\nabla \mathbf{u}_{m,n}\| \|\nabla \psi_{m,n}\| + \frac{\epsilon_1}{6} |\nabla \psi_d^{m,n}|^2 \\ + \frac{3c_4^2}{2\epsilon_1} \|\psi_{m,n}\|_{H^2(\Omega)} \|\nabla \psi_{m,n}\|_{H^2(\Omega)} + \frac{\epsilon_1}{6} |\nabla \psi_d^{m,n}|^2.$$

Let $c_5 = 3c_3^2/2\epsilon_1$ and $c_6 = 3c_4^2/2\epsilon_1$, we have

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_d^{m,n})| \leq c_5 \|\nabla \mathbf{u}_{m,n}\| \|\nabla \psi_{m,n}\| + \frac{\epsilon_1}{3} |\nabla \psi_d^{m,n}|^2 \\ + c_6 \|\psi_{m,n}\|_{H^2(\Omega)} \|\nabla \psi_{m,n}\|_{H^2(\Omega)},$$

and using elliptic estimate, we have

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_d^{m,n})| \leq c_5 \|\nabla \mathbf{u}_{m,n}\| \|\nabla \psi_{m,n}\| + \frac{\epsilon_1}{3} |\nabla \psi_d^{m,n}|^2 \\ + c_7 \|\psi_{m,n}\|_{H^2(\Omega)} (|\nabla \psi_{m,n}| + |\nabla \psi_d^{m,n}|),$$

or

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_d^{m,n})| \leq c_5 \|\nabla \mathbf{u}_{m,n}\| \|\nabla \psi_{m,n}\| + \frac{\epsilon_1}{3} |\nabla \psi_d^{m,n}|^2 \\ + c_7 \|\psi_{m,n}\|_{H^2(\Omega)} |\nabla \psi_{m,n}| \\ + c_7 \|\psi_{m,n}\|_{H^2(\Omega)} |\nabla \psi_d^{m,n}|.$$

By the Young's inequality, we obtain

$$|2b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \Delta\psi_d^{m,n})| \leq \frac{c_5^2}{2} \|\nabla \mathbf{u}_{m,n}\|^2 + \frac{1}{2} \|\nabla \psi_{m,n}\|^2 + \frac{\epsilon_1}{2} |\nabla \psi_d^{m,n}|^2 \\ + \frac{c_7^2}{2} \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{1}{2} |\nabla \psi_{m,n}|^2 \\ + \frac{3c_7^2}{2\epsilon_1} \|\psi_{m,n}\|_{H^2(\Omega)}^2, \quad (2.85)$$

Consider again the following integral

$$I = 2 \int_{\Omega} |\nabla A_2(\psi_{m,n}, c_{m,n})|_2 |\nabla \psi_d^{m,n}|_2 d\mathbf{x},$$

and according to the Lemma 3, the above equation takes the form

$$I \leq 2c_9 \int_{\Omega} (1 + |\nabla c_{m,n}|_2 + |\nabla \psi_{m,n}|_2) |\nabla \psi_d^{m,n}|_2 d\mathbf{x}.$$

By using Hölder's and then Young's inequality, we get

$$I \leq c_{10} (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) + \frac{\epsilon_1}{2} |\nabla \psi_d^{m,n}|^2, \quad (2.86)$$

and employing the inequalities (2.85) and (2.86) in the relation (2.84), we arrive at

$$\begin{aligned} \frac{d}{dt} |\psi_d^{m,n}|^2 + 2\epsilon_1 |\nabla \psi_d^{m,n}|^2 &\leq c_{10} (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) + \epsilon_1 |\nabla \psi_d^{m,n}|^2 \\ &\quad + \frac{c_5^2}{2} \|\nabla \mathbf{u}_{m,n}\|^2 + \frac{1}{2} \|\nabla \psi_{m,n}\|^2 + c_8 \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{1}{2} |\nabla \psi_{m,n}|^2. \end{aligned}$$

Simplification of above inequality gives

$$\begin{aligned} \frac{d}{dt} |\psi_d^{m,n}|^2 + \epsilon_1 |\nabla \psi_d^{m,n}|^2 &\leq c_{10} (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) \\ &\quad + \frac{1}{2} \|\nabla \psi_{m,n}\|^2 + \frac{c_5^2}{2} \|\nabla \mathbf{u}_{m,n}\|^2 \\ &\quad + c_8 \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{1}{2} |\nabla \psi_{m,n}|^2. \end{aligned} \quad (2.87)$$

From the above inequality (2.87), we can deduce that

$$\begin{aligned} \frac{d}{dt} |\psi_d^{m,n}|^2 &\leq c_{10} (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) + \frac{1}{2} \|\nabla \psi_{m,n}\|^2 \\ &\quad + \frac{c_5^2}{2} \|\nabla \mathbf{u}_{m,n}\|^2 + c_8 \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{1}{2} |\nabla \psi_{m,n}|^2. \end{aligned}$$

Integrating above inequality over $(0, t)$ for $t \in (0, T_f)$, we have

$$\begin{aligned} \int_0^t \frac{d}{ds} |\psi_d^{m,n}|^2 ds &\leq c_{10} \int_0^t (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) d\tau \\ &\quad + \frac{1}{2} \int_0^t \|\nabla \psi_{m,n}\|^2 d\tau + \frac{c_5^2}{2} \int_0^t \|\nabla \mathbf{u}_{m,n}\|^2 d\tau \\ &\quad + c_8 \int_0^t \|\psi_{m,n}\|_{H^2(\Omega)}^2 d\tau + \frac{1}{2} \int_0^t |\nabla \psi_{m,n}|^2 d\tau. \end{aligned}$$

The results (2.39), (2.59) and (2.72) reduces the above inequality to the following form

$$\int_0^t \frac{d}{ds} |\psi_d^{m,n}(s)|^2 ds \leq c_{11}, \quad \forall t \in (0, T_f)$$

and

$$|\psi_d^{m,n}(t)|^2 \leq c_{11} + |\psi_d^{m,n}(0)|^2, \quad \forall t \in (0, T_f).$$

Making use of the relation (2.21) and since $\psi_0 \in H_0^2(\Omega)$, we have

$$|\psi_d^{m,n}(t)|^2 \leq c_{12}, \quad \forall t \in (0, T_f).$$

From the above inequality, we conclude that

$$\|\psi_d^{m,n}(t)\|_{L^\infty(0, T_f; L^2(\Omega))} \leq c_{12}. \quad (2.88)$$

This shows that $\psi_d^{m,n} = \Delta\psi_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; L^2(\Omega))$. From equation (2.71) and (2.88), we conclude that $\psi_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; H^2(\Omega))$. This implies that

$$\|\psi_{m,n}\|_{L^\infty(0, T_f; H^2(\Omega))} \leq c_{13}. \quad (2.89)$$

Again by integrating the inequality (2.87) over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\begin{aligned} \int_0^t \frac{d}{ds} |\psi_d^{m,n}|^2 ds + \epsilon_1 \int_0^t |\nabla \psi_d^{m,n}|^2 ds &\leq c_{10} \int_0^t (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) ds \\ &\quad + \frac{1}{2} \int_0^t \|\nabla \psi_{m,n}\|^2 ds + \frac{c_5^2}{2} \int_0^t \|\nabla \mathbf{u}_{m,n}\|^2 ds \\ &\quad + c_8 \int_0^t \|\psi_{m,n}\|_{H^2(\Omega)}^2 ds + \frac{1}{2} \int_0^t |\nabla \psi_{m,n}|^2 ds, \end{aligned}$$

or

$$\begin{aligned} |\psi_d^{m,n}(t)|^2 + \epsilon_1 \int_0^t |\nabla \psi_d^{m,n}|^2 ds &\leq c_{10} \int_0^t (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) ds \\ &\quad + \frac{1}{2} \int_0^t \|\nabla \psi_{m,n}\|^2 ds + \frac{c_5^2}{2} \int_0^t \|\nabla \mathbf{u}_{m,n}\|^2 ds + \\ &\quad c_8 \int_0^t \|\psi_{m,n}\|_{H^2(\Omega)}^2 ds + \frac{1}{2} \int_0^t |\nabla \psi_{m,n}|^2 ds + |\psi_d^{m,n}(0)|^2. \end{aligned}$$

From the above inequality, we can deduce that

$$\begin{aligned} \epsilon_1 \int_0^t |\nabla \psi_d^{m,n}|^2 ds &\leq c_{10} \int_0^t (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) ds + \frac{1}{2} \int_0^t \|\nabla \psi_{m,n}\|^2 ds \\ &\quad + \frac{c_5^2}{2} \int_0^t \|\nabla \mathbf{u}_{m,n}\|^2 ds + c_8 \int_0^t \|\psi_{m,n}\|_{H^2(\Omega)}^2 ds \\ &\quad + \frac{1}{2} \int_0^t |\nabla \psi_{m,n}|^2 ds + |\psi_d^{m,n}(0)|^2, \end{aligned}$$

and according to the relations (2.89), (2.71), (2.38) and as $\psi_0 \in H_0^2(\Omega)$, we conclude that

$$\int_0^t |\nabla \psi_d^{m,n}(s)|^2 ds \leq c_{14}, \quad \forall t \in (0, T_f).$$

This implies that

$$\|\nabla \psi_d^{m,n}(s)\|_{L^2(0,T_f;L^2(\Omega))} \leq c_{14}.$$

This shows that $\nabla \psi_d^{m,n}$ is uniformly bounded in $L^2(0,T_f;L^2(\Omega))$. Therefore using equation (2.72), (2.89) and the above inequality, we can conclude that $\psi_{m,n}$ is uniformly bounded in $L^\infty(0,T_f,H^2(\Omega)) \cap L^2(0,T_f;H^3(\Omega))$. That implies

$$\|\psi_{m,n}\|_{L^\infty(0,T_f,H^2(\Omega)) \cap L^2(0,T_f;H^3(\Omega))} \leq c_{15}. \quad (2.90)$$

Now multiplying equation (2.76) by $\partial \psi_k^{m,n}/\partial t$ on both sides and then taking sum over k , where $k = 1, 2, \dots, n$, we obtain by using relation (2.27)

$$\begin{aligned} - \left(\frac{\partial \psi_{m,n}}{\partial t}, \frac{\partial \psi_d^{m,n}}{\partial t} \right) - \epsilon_1 \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} - b_\psi(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t}) \\ = \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x}. \end{aligned} \quad (2.91)$$

Consider the following integral

$$\begin{aligned} \int_{\Omega} \operatorname{div} \left(\frac{\partial \psi_{m,n}}{\partial t} \frac{\partial \nabla \psi_{m,n}}{\partial t} \right) d\mathbf{x} &= \int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \frac{\partial \Delta \psi_{m,n}}{\partial t} d\mathbf{x} \\ &+ \int_{\Omega} \nabla \left(\frac{\partial \psi_{m,n}}{\partial t} \right) \frac{\partial \nabla \psi_{m,n}}{\partial t} d\mathbf{x}. \end{aligned}$$

By using divergence theorem, we have

$$\int_{\Gamma} \frac{\partial \psi_{m,n}}{\partial t} \frac{\partial \nabla \psi_{m,n}}{\partial t} \cdot \mathbf{n} d\Gamma = \int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} + \int_{\Omega} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|_2^2 d\mathbf{x}.$$

Since $\frac{\partial \nabla \psi_{m,n}}{\partial t} \cdot \mathbf{n} = 0$ on the boundary Γ , therefore we have

$$\int_{\Omega} \frac{\partial \psi_{m,n}}{\partial t} \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} = - \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|_2^2. \quad (2.92)$$

Again consider the integral

$$\int_{\Omega} \operatorname{div} \left(\nabla \psi_{m,n} \frac{\partial \psi_d^{m,n}}{\partial t} \right) d\mathbf{x} = \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \left(\frac{\partial \psi_d^{m,n}}{\partial t} \right) d\mathbf{x} + \int_{\Omega} \Delta \psi_{m,n} \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x},$$

and by the divergence theorem, we have (since $\Delta \psi_{m,n} = \psi_d^{m,n}$)

$$\int_{\Gamma} \nabla \psi_{m,n} \frac{\partial \psi_d^{m,n}}{\partial t} \cdot \mathbf{n} d\Gamma = \int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \left(\frac{\partial \psi_d^{m,n}}{\partial t} \right) d\mathbf{x} + \int_{\Omega} \psi_d^{m,n} \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x}.$$

As we know that $\nabla \psi_{m,n} \cdot \mathbf{n} = 0$ on the boundary Γ , therefore we have

$$\int_{\Omega} \nabla \psi_{m,n} \cdot \nabla \left(\frac{\partial \psi_d^{m,n}}{\partial t} \right) d\mathbf{x} = - \frac{1}{2} \frac{d}{dt} |\psi_d^{m,n}|^2. \quad (2.93)$$

Consider now the following integral

$$\begin{aligned} \int_{\Omega} \operatorname{div} \left(A_2(\psi_{m,n}, c_{m,n}) \frac{\partial \nabla \psi_{m,n}}{\partial t} \right) d\mathbf{x} &= \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} \\ &+ \int_{\Omega} \nabla A_2(\psi_{m,n}, c_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x}, \end{aligned}$$

and employing the divergence theorem, we have

$$\begin{aligned} \int_{\Gamma} A_2(\psi_{m,n}, c_{m,n}) \frac{\partial \nabla \psi_{m,n}}{\partial t} \cdot \mathbf{n} d\Gamma &= \int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} \\ &+ \int_{\Omega} \nabla A_2(\psi_{m,n}, c_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x}. \end{aligned}$$

Since $\frac{\partial \nabla \psi_{m,n}}{\partial t} \cdot \mathbf{n} = 0$ on the boundary Γ , therefore we have

$$\int_{\Omega} A_2(\psi_{m,n}, c_{m,n}) \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} = - \int_{\Omega} \nabla A_2(\psi_{m,n}, c_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x}. \quad (2.94)$$

Making use of equations (2.92)-(2.94) in equation (2.91), we obtain

$$\begin{aligned} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \frac{\epsilon_1}{2} \frac{d}{dt} |\psi_d^{m,n}|^2 &= b_{\psi} \left(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t} \right) \\ &- \int_{\Omega} \nabla A_2(\psi_{m,n}, c_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x}, \end{aligned}$$

or

$$\begin{aligned} 2 \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \epsilon_1 \frac{d}{dt} |\psi_d^{m,n}|^2 &\leq \left| 2b_{\psi} \left(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t} \right) \right| \\ &+ 2 \int_{\Omega} |\nabla A_2(\psi_{m,n}, c_{m,n})|_2 \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|_2 d\mathbf{x}. \end{aligned} \quad (2.95)$$

Consider the following integral

$$\begin{aligned} \int_{\Omega} \operatorname{div} \left((\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} \right) d\mathbf{x} &= \int_{\Omega} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} \\ &+ \int_{\Omega} \nabla (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x}, \end{aligned}$$

Making use of the divergence theorem, we have

$$\begin{aligned} \int_{\Gamma} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} \cdot \mathbf{n} d\Gamma &= \int_{\Omega} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} \\ &+ \int_{\Omega} \nabla (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x}. \end{aligned}$$

As $\nabla\psi_{m,n} \cdot \mathbf{n} = 0$ on the boundary Γ , therefore we obtain

$$\int_{\Omega} (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \frac{\partial \psi_d^{m,n}}{\partial t} d\mathbf{x} = - \int_{\Omega} \nabla (\mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x},$$

or

$$\begin{aligned} b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t}) &= - \int_{\Omega} (\nabla \mathbf{u}_{m,n} \cdot \nabla \psi_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x} \\ &\quad + \int_{\Omega} (\nabla(\nabla \psi_{m,n}) \cdot \mathbf{u}_{m,n}) \cdot \nabla \frac{\partial \psi_{m,n}}{\partial t} d\mathbf{x}, \end{aligned}$$

and then

$$\begin{aligned} \left| 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t}) \right| &\leq 2 \int_{\Omega} |\nabla \mathbf{u}_{m,n}|_2 |\nabla \psi_{m,n}|_2 \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|_2 d\mathbf{x} \\ &\quad + 2 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\nabla(\nabla \psi_{m,n})|_2 \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|_2 d\mathbf{x}, \end{aligned}$$

and with the help of the Hölder's inequality, we have

$$\begin{aligned} \left| 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t}) \right| &\leq 2 \|\nabla \mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right| \\ &\quad + 2 \|\mathbf{u}_{m,n}\|_{(L^4(\Omega))^2} \|\nabla(\nabla \psi_{m,n})\|_{L^4(\Omega)} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|. \end{aligned}$$

According to the Lemma 1, the above inequality takes the form

$$\begin{aligned} \left| 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t}) \right| &\leq c_{16} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)} \|\psi_{m,n}\|_{H^2(\Omega)} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right| \\ &\quad + c_{17} \|\mathbf{u}_{m,n}\| \|\nabla(\nabla \psi_{m,n})\| \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|. \end{aligned}$$

Using Young's inequality, we arrive at

$$\begin{aligned} \left| 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t}) \right| &\leq \frac{5c_{16}^2}{4} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 \|\psi_{m,n}\|_{H^2(\Omega)}^2 \\ &\quad + \frac{2}{5} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \frac{5c_{17}^2}{4} \|\mathbf{u}_{m,n}\|^2 \|\nabla(\nabla \psi_{m,n})\|^2, \end{aligned}$$

and by the definition of Sobolev norm and the results (2.57) and (2.90), the above inequality takes the form

$$\begin{aligned} \left| 2b_{\psi}(\mathbf{u}_{m,n}, \psi_{m,n}, \frac{\partial \psi_d^{m,n}}{\partial t}) \right|_2 &\leq c_{18} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 \\ &\quad + \frac{2}{5} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + c_{19} \|\nabla \psi_{m,n}\|_{H^2(\Omega)}^2. \end{aligned} \tag{2.96}$$

Consider the following integral

$$I_1 = 2 \int_{\Omega} |\nabla A_2(\psi_{m,n}, c_{m,n})|_2 \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|_2 d\mathbf{x},$$

According to the Lemma 3, we have

$$I_1 \leq 2c_{20} \int_{\Omega} (1 + |\nabla c_{m,n}|_2 + |\nabla \psi_{m,n}|_2) \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|_2 d\mathbf{x},$$

and using Hölder's inequality, we get

$$I_1 \leq 2c_{20} (1 + |\nabla c_{m,n}| + |\nabla \psi_{m,n}|) \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|.$$

Applying the Young's inequality, the above inequality takes the form

$$I_1 \leq 5c_{20}^2 (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2) + \frac{3}{5} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2. \quad (2.97)$$

Substitution of the relations (2.96) and (2.97) in the inequality (2.95) gives

$$\begin{aligned} 2 \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \epsilon_1 \frac{d}{dt} |\psi_d^{m,n}|^2 &\leq c_{18} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 \\ &\quad + \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + c_{19} \|\nabla \psi_{m,n}\|_{H^2(\Omega)}^2 \\ &\quad + 5c_{20}^2 (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2), \end{aligned}$$

or

$$\begin{aligned} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \epsilon_1 \frac{d}{dt} |\psi_d^{m,n}|^2 &\leq c_{18} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 + c_{19} \|\nabla \psi_{m,n}\|_{H^2(\Omega)}^2 \\ &\quad + 5c_{20}^2 (1 + |\nabla c_{m,n}|^2 + |\nabla \psi_{m,n}|^2). \end{aligned}$$

The relation (2.90) reduces the above inequality to

$$\begin{aligned} \left| \nabla \frac{\partial \psi_{m,n}}{\partial t} \right|^2 + \epsilon_1 \frac{d}{dt} |\psi_d^{m,n}|^2 &\leq c_{18} \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 + c_{19} \|\nabla \psi_{m,n}\|_{H^2(\Omega)}^2 \\ &\quad + c_{21} (1 + |\nabla c_{m,n}|^2). \end{aligned}$$

Integrating over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\begin{aligned} \int_0^t \left| \nabla \frac{\partial \psi_{m,n}}{\partial s} \right|^2 ds + \epsilon_1 |\psi_d^{m,n}(t)|^2 &\leq c_{18} \int_0^t \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 ds + |\psi_d^{m,n}(0)|^2 \\ &\quad + c_{19} \int_0^t \|\nabla \psi_{m,n}\|_{H^2(\Omega)}^2 ds + c_{21} \int_0^t (1 + |\nabla c_{m,n}|^2) ds. \end{aligned}$$

From above inequality, we can deduce that

$$\begin{aligned} \int_0^t \left| \nabla \frac{\partial \psi_{m,n}}{\partial s} \right|^2 ds &\leq c_{18} \int_0^t \|\mathbf{u}_{m,n}\|_{H^2(\Omega)}^2 ds + |\psi_d^{m,n}(0)|^2 \\ &\quad + c_{19} \int_0^t \|\nabla \psi_{m,n}\|_{H^2(\Omega)}^2 ds + c_{21} \int_0^t (1 + |\nabla c_{m,n}|^2) ds. \end{aligned}$$

As $\psi \in H_0^2(\Omega)$ and using the relations (2.39), (2.59) and (2.90), we have

$$\int_0^{T_f} \left| \nabla \frac{\partial \psi_{m,n}(s)}{\partial s} \right|^2 ds \leq c_{22},$$

and this implies that

$$\left\| \nabla \frac{\partial \psi_{m,n}(t)}{\partial t} \right\|_{L^2(0, T_f; L^2(\Omega))} \leq c_{22}. \quad (2.98)$$

This shows that $\nabla \left(\frac{\partial \psi_{m,n}(t)}{\partial t} \right)$ is uniformly bounded in $L^2(0, T_f; L^2(\Omega))$. As we have already seen that $\psi_{m,n}$ is uniformly bounded in $H^1(0, T_f; L^2(\Omega))$, and now by equation (2.98), we can conclude that $\psi_{m,n}$ is uniformly bounded in $H^1(0, T_f; H^1(\Omega))$. That is

$$\|\psi_{m,n}\|_{H^1(0, T_f; H^1(\Omega))} \leq c_{23}. \quad (2.99)$$

Multiplying now equation (2.24) on both sides by μ_k and then using relations (2.13) and (2.27), we have

$$\begin{aligned} - \left(\frac{\partial c_{m,n}}{\partial t}, \Delta c_{m,n} \right) - b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n}) - \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla (\Delta c_{m,n}) d\mathbf{x} \\ - \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla (\Delta c_{m,n}) d\mathbf{x} = 0. \end{aligned}$$

Employing Green's formula and as $\nabla c_{m,n} \cdot \mathbf{n} = 0$, the above equation takes the form

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\nabla c_{m,n}|^2 - b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n}) + \int_{\Omega} \operatorname{div} (D(\psi_{m,n}) \nabla c_{m,n}) \Delta c_{m,n} d\mathbf{x} \\ + \int_{\Omega} \operatorname{div} (A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n}) \Delta c_{m,n} d\mathbf{x} = 0. \end{aligned}$$

Simplifying above equation, we have

$$\begin{aligned} \frac{d}{dt} |\nabla c_{m,n}|^2 + 2 \int_{\Omega} D(\psi_{m,n}) |\Delta c_{m,n}|^2 d\mathbf{x} = 2b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n}) \\ - 2 \int_{\Omega} \nabla D(\psi_{m,n}) \cdot \nabla c_{m,n} \Delta c_{m,n} d\mathbf{x} - 2 \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \Delta \psi_{m,n} \Delta c_{m,n} d\mathbf{x} \\ - 2 \int_{\Omega} \nabla A_3(\psi_{m,n}, c_{m,n}) \cdot \nabla \psi_{m,n} \Delta c_{m,n} d\mathbf{x}. \end{aligned} \quad (2.100)$$

Consider

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq 2 \int_{\Omega} |\mathbf{u}_{m,n}|_2 |\nabla c_{m,n}|_2 |\Delta c_{m,n}|_2 d\mathbf{x},$$

and with the help of the Hölder's inequality, we have

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq 2 \|\mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla c_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}|.$$

Using Gagliardo-Nirenberg's inequality (see Lemma 1), we have

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq 2c_{24} \|\mathbf{u}_{m,n}\| |\nabla c_{m,n}|^{\frac{1}{2}} \|\nabla c_{m,n}\|^{\frac{1}{2}} |\Delta c_{m,n}|,$$

and then applying Young's inequality, the above inequality takes the form

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq \frac{4c_{24}^2}{D_0} |\mathbf{u}_{m,n}|^2 \|\mathbf{u}_{m,n}\| |\nabla c_{m,n}| \|\nabla c_{m,n}\| + \frac{D_0}{4} |\Delta c_{m,n}|^2.$$

According to the relation (2.57), we obtain

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq c_{25} |\nabla c_{m,n}| \|\nabla c_{m,n}\| + \frac{D_0}{4} |\Delta c_{m,n}|^2.$$

Since $\|\nabla c_{m,n}\| \leq c_{26} \|c_{m,n}\|_{H^2(\Omega)}$, thus we have

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq c_{27} |\nabla c_{m,n}| \|c_{m,n}\|_{H^2(\Omega)} + \frac{D_0}{4} |\Delta c_{m,n}|^2,$$

and using elliptic estimate, the above inequality takes the form

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq c_{28} |\nabla c_{m,n}| (|\Delta c_{m,n}| + |c_{m,n}|) + \frac{D_0}{4} |\Delta c_{m,n}|^2,$$

or

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq c_{28} |\nabla c_{m,n}| |\Delta c_{m,n}| + c_{28} |c_{m,n}| |\nabla c_{m,n}| + \frac{D_0}{4} |\Delta c_{m,n}|^2.$$

By the Young's inequality, we have

$$\begin{aligned} 2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| &\leq \frac{c_{28}^2}{D_0} |\nabla c_{m,n}|^2 + \frac{D_0}{4} |\Delta c_{m,n}|^2 + \frac{c_{28}^2}{2} |\nabla c_{m,n}|^2 \\ &\quad + \frac{1}{2} |c_{m,n}|^2 + \frac{D_0}{4} |\Delta c_{m,n}|^2. \end{aligned}$$

The above inequality can be simplified as

$$2 |b_c(\mathbf{u}_{m,n}, c_{m,n}, \Delta c_{m,n})| \leq c_{29} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2 + \frac{D_0}{2} |\Delta c_{m,n}|^2. \quad (2.101)$$

Making use of the relation (2.101) in the inequality (2.100), we have

$$\begin{aligned} \frac{d}{dt} |\nabla c_{m,n}|^2 + 2 \int_{\Omega} D(\psi_{m,n}) |\Delta c_{m,n}|^2 d\mathbf{x} &\leq c_{29} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2 \\ &+ \frac{D_0}{2} |\Delta c_{m,n}|^2 + 2 \int_{\Omega} |\nabla D(\psi_{m,n})|_2 |\nabla c_{m,n}|_2 |\Delta c_{m,n}| d\mathbf{x} \\ &+ 2 \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) |\Delta \psi_{m,n}|_2 |\Delta c_{m,n}| d\mathbf{x} \\ &+ 2 \int_{\Omega} |\nabla A_3(\psi_{m,n}, c_{m,n})|_2 |\nabla \psi_{m,n}|_2 |\Delta c_{m,n}| d\mathbf{x}, \end{aligned}$$

Using hypothesis (H1) – (H5) and the Lemma 3, we arrive at

$$\begin{aligned} \frac{d}{dt} |\nabla c_{m,n}|^2 + 2D_0 \int_{\Omega} |\Delta c_{m,n}|^2 d\mathbf{x} &\leq c_{29} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2 \\ &+ \frac{D_0}{2} |\Delta c_{m,n}|^2 + 2c_{30} \int_{\Omega} (1 + |\nabla \psi_{m,n}|_2) |\nabla c_{m,n}|_2 |\Delta c_{m,n}| d\mathbf{x} \\ &+ 2a_3 \int_{\Omega} |\Delta \psi_{m,n}|_2 |\Delta c_{m,n}| d\mathbf{x} \\ &+ 2c_{31} \int_{\Omega} (1 + |\nabla \psi_{m,n}|_2 + |\nabla c_{m,n}|_2) |\nabla \psi_{m,n}|_2 |\Delta c_{m,n}| d\mathbf{x}. \end{aligned}$$

Applying the Hölder's inequality, we arrive at

$$\begin{aligned} \frac{d}{dt} |\nabla c_{m,n}|^2 + 2D_0 |\Delta c_{m,n}|^2 &\leq c_{29} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2 \\ &+ \frac{D_0}{2} |\Delta c_{m,n}|^2 + 2c_{30} |\nabla c_{m,n}| |\Delta c_{m,n}| + 2a_3 |\Delta \psi_{m,n}| |\Delta c_{m,n}| \\ &+ 2c_{31} |\nabla \psi_{m,n}| |\Delta c_{m,n}| + 2c_{31} \|\nabla \psi_{m,n}\|_{L^4(\Omega)}^2 |\Delta c_{m,n}| \\ &+ c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}|, \end{aligned}$$

and further by the Young's inequality, we have

$$\begin{aligned} \frac{d}{dt} |\nabla c_{m,n}|^2 + 2D_0 |\Delta c_{m,n}|^2 &\leq c_{29} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2 \\ &+ \frac{D_0}{2} |\Delta c_{m,n}|^2 + \frac{12c_{30}^2}{D_0} |\nabla c_{m,n}|^2 + \frac{D_0}{12} |\Delta c_{m,n}|^2 \\ &+ \frac{12a_3^2}{D_0} |\Delta \psi_{m,n}|^2 + \frac{D_0}{12} |\Delta c_{m,n}|^2 + \frac{12c_{31}^2}{D_0} |\nabla \psi_{m,n}|^2 \\ &+ \frac{D_0}{12} |\Delta c_{m,n}|^2 + 2c_{31} \|\nabla \psi_{m,n}\|_{L^4(\Omega)}^2 |\Delta c_{m,n}| \\ &+ c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}|. \end{aligned}$$

As $\|\nabla\psi_{m,n}\|_{L^4(\Omega)} \leq c_{33} \|\psi_{m,n}\|_{H^2(\Omega)}$, then the above inequality takes the form

$$\begin{aligned} \frac{d}{dt} |\nabla c_{m,n}|^2 + 2D_0 |\Delta c_{m,n}|^2 &\leq \frac{3D_0}{4} |\Delta c_{m,n}|^2 + c_{33} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2 \\ &\quad + \frac{12a_3^2}{D_0} |\Delta\psi_{m,n}|^2 + 12c_{31}^2 |\nabla\psi_{m,n}|^2 + 2c_{31} \|\psi_{m,n}\|_{H^2(\Omega)}^2 |\Delta c_{m,n}| \\ &\quad + c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\psi_{m,n}\|_{H^2(\Omega)} |\Delta c_{m,n}|, \end{aligned} \quad (2.102)$$

and using Gagliardo-Nirenberg's inequality in the last term of the above inequality (2.102) we have

$$\begin{aligned} c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\psi_{m,n}\|_{H^2(\Omega)} |\Delta c_{m,n}| &\leq \\ c_{34} \|\psi_{m,n}\|_{H^2(\Omega)} |\nabla c_{m,n}|^{\frac{1}{2}} \|\nabla c_{m,n}\|^{\frac{1}{2}} |\Delta c_{m,n}|, \end{aligned}$$

As $\|\nabla c_{m,n}\| \leq c_{35} \|c_{m,n}\|_{H^2(\Omega)}$ and with the help of Young's inequality, the above inequality becomes

$$\begin{aligned} c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\nabla\psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| &\leq \frac{D_0}{12} |\Delta c_{m,n}|^2 \\ &\quad + c_{36} \|\psi_{m,n}\|_{H^2(\Omega)}^2 |\nabla c_{m,n}| \|c_{m,n}\|_{H^2(\Omega)}. \end{aligned}$$

Making use of the relation (2.89), we get

$$\begin{aligned} c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\nabla\psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| &\leq \frac{D_0}{12} |\Delta c_{m,n}|^2 \\ &\quad + c_{37} |\nabla c_{m,n}| \|c_{m,n}\|_{H^2(\Omega)}. \end{aligned}$$

According to the elliptic estimate, the above inequality becomes

$$\begin{aligned} c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\nabla\psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| &\leq \frac{D_0}{12} |\Delta c_{m,n}|^2 \\ &\quad + c_{38} |\nabla c_{m,n}| (|\Delta c_{m,n}| + |c_{m,n}|), \end{aligned}$$

and then by the Young's inequality, we have

$$\begin{aligned} c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\nabla\psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| &\leq \frac{D_0}{12} |\Delta c_{m,n}|^2 \\ &\quad + \frac{3c_{38}^2}{D_0} |\nabla c_{m,n}|^2 + \frac{D_0}{12} |\Delta c_{m,n}|^2 \\ &\quad + \frac{c_{38}^2}{2} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2. \end{aligned}$$

Simplifying the above inequality, we have

$$\begin{aligned} c_{32} \|\nabla c_{m,n}\|_{L^4(\Omega)} \|\nabla\psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| &\leq \frac{D_0}{6} |\Delta c_{m,n}|^2 \\ &\quad + c_{39} |\nabla c_{m,n}|^2 + \frac{1}{2} |c_{m,n}|^2. \end{aligned} \quad (2.103)$$

Using Gagliardo-Nirenberg's inequality, we have

$$2c_{31} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| \leq c_{40} |\nabla \psi_{m,n}|^{\frac{1}{2}} \|\nabla \psi_{m,n}\|^{\frac{1}{2}} |\Delta c_{m,n}|.$$

Applying the Young's inequality and using the relation (2.89), the above inequality takes the form

$$2c_{31} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| \leq c_{41} \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{D_0}{12} |\Delta c_{m,n}|^2.$$

As we know that $|\nabla \psi_{m,n}| \leq c_{42} \|\psi_{m,n}\|_{H^2(\Omega)}$ and $\|\nabla \psi_{m,n}\| \leq \|\psi_{m,n}\|_{H^2(\Omega)}$, thus we have

$$2c_{31} \|\nabla \psi_{m,n}\|_{L^4(\Omega)} |\Delta c_{m,n}| \leq c_{43} \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{D_0}{12} |\Delta c_{m,n}|^2. \quad (2.104)$$

Using the inequalities (2.103) and (2.104) in the inequality (2.102), we obtain

$$\begin{aligned} \frac{d}{dt} |\nabla c_{m,n}|^2 + 2D_0 |\Delta c_{m,n}|^2 &\leq D_0 |\Delta c_{m,n}|^2 + c_{33} |\nabla c_{m,n}|^2 + |c_{m,n}|^2 \\ &\quad + \frac{12a_3^2}{D_0} |\Delta \psi_{m,n}|^2 + \frac{12c_{31}^2}{D_0} |\nabla \psi_{m,n}|^2 \\ &\quad + \frac{12c_{43}^2}{D_0} \|\psi_{m,n}\|_{H^2(\Omega)}^2 + c_{39} |\nabla c_{m,n}|^2. \end{aligned}$$

Since $|\nabla \psi_{m,n}| \leq c_{44} \|\psi_{m,n}\|_{H^2(\Omega)}$ and $|\Delta \psi_{m,n}| \leq c_{45} \|\psi_{m,n}\|_{H^2(\Omega)}$, thus we have

$$\frac{d}{dt} |\nabla c_{m,n}|^2 + D_0 |\Delta c_{m,n}|^2 \leq c_{46} |\nabla c_{m,n}|^2 + |c_{m,n}|^2 + c_{47} \|\psi_{m,n}\|_{H^2(\Omega)}^2,$$

According to the relations (2.39) and (2.89), the above inequality takes the form

$$\frac{d}{dt} |\nabla c_{m,n}|^2 + D_0 |\Delta c_{m,n}|^2 \leq c_{48} (1 + |\nabla c_{m,n}|^2). \quad (2.105)$$

From the above inequality, we can deduce that

$$\frac{d}{dt} |\nabla c_{m,n}|^2 \leq c_{48} (1 + |\nabla c_{m,n}|^2),$$

and with the help of Gronwall's lemma and the relation (2.19), we can easily get

$$|\nabla c_{m,n}(t)|^2 \leq c_{49}, \quad \forall t \in (0, T_f)$$

this implies that

$$\|\nabla c_{m,n}\|_{L^\infty(0, T_f; L^2(\Omega))} \leq c_{49}. \quad (2.106)$$

This shows that $\nabla c_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; L^2(\Omega))$. Thus from relations (2.106) and (2.39), we can deduce that $c_{m,n}$ is uniformly bounded in $L^\infty(0, T_f; H^1(\Omega))$. i.e.,

$$\|c_{m,n}\|_{L^\infty(0, T_f; H^1(\Omega))} \leq c_{50}. \quad (2.107)$$

Now integrating the inequality (2.105) over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\int_0^t \frac{d}{ds} |\nabla c_{m,n}|^2 ds + D_0 \int_0^t |\Delta c_{m,n}|^2 ds \leq c_{48} \int_0^t (1 + |\nabla c_{m,n}|^2) ds,$$

or

$$|\nabla c_{m,n}(t)|^2 + D_0 \int_0^t |\Delta c_{m,n}|^2 ds \leq c_{48} \int_0^t (1 + |\nabla c_{m,n}|^2) ds + |\nabla c_{m,n}(0)|^2.$$

From the above inequality, we can deduce that

$$D_0 \int_0^t |\Delta c_{m,n}|^2 ds \leq c_{48} \int_0^t (1 + |\nabla c_{m,n}|^2) ds + |\nabla c_{m,n}(0)|^2,$$

Making use of (2.107) and as $c_0 \in H^1(\Omega)$, we can easily arrive at

$$D_0 \int_0^t |\Delta c_{m,n}|^2 ds \leq c_{51}, \quad \forall t \in (0, T_f).$$

Thus we have

$$\|\Delta c_{m,n}\|_{L^2(0, T_f; L^2(\Omega))} \leq c_{52}. \quad (2.108)$$

This shows that $\Delta c_{m,n}$ is uniformly bounded in $L^2(0, T_f; L^2(\Omega))$. Using the results (2.107), (2.108) and elliptic estimate, we can deduce that $c_{m,n}$ is uniformly bounded in $L^2(0, T_f; H^2(\Omega))$, that is

$$\|c_{m,n}\|_{L^2(0, T_f; H^2(\Omega))} \leq c_{53}. \quad (2.109)$$

Now multiplying equation (2.24) by $\partial c_k^{m,n} / \partial t$ on both sides and then taking sum over k , where $k = 1, 2, \dots, n$, we obtain

$$\begin{aligned} \left(\frac{\partial c_{m,n}}{\partial t}, \frac{\partial c_{m,n}}{\partial t} \right) + b_c(\mathbf{u}_{m,n}, c_{m,n}, \frac{\partial c_{m,n}}{\partial t}) + \int_{\Omega} D(\psi_{m,n}) \nabla c_{m,n} \cdot \nabla \left(\frac{\partial c_{m,n}}{\partial t} \right) d\mathbf{x} \\ + \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \nabla \psi_{m,n} \cdot \nabla \left(\frac{\partial c_{m,n}}{\partial t} \right) d\mathbf{x} = 0, \end{aligned}$$

and by applying Green's formula and as $\nabla c_{m,n} \cdot \mathbf{n} = 0$ on Γ , the above equation takes the form

$$\begin{aligned} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 + b_c(\mathbf{u}_{m,n}, c_{m,n}, \frac{\partial c_{m,n}}{\partial t}) = \int_{\Omega} D(\psi_{m,n}) \Delta c_{m,n} \frac{\partial c_{m,n}}{\partial t} d\mathbf{x} \\ + \int_{\Omega} \nabla D(\psi_{m,n}) \cdot \nabla c_{m,n} \frac{\partial c_{m,n}}{\partial t} d\mathbf{x} + \int_{\Omega} A_3(\psi_{m,n}, c_{m,n}) \Delta \psi_{m,n} \frac{\partial c_{m,n}}{\partial t} d\mathbf{x} \\ + \int_{\Omega} \nabla A_3(\psi_{m,n}, c_{m,n}) \cdot \nabla \psi_{m,n} \frac{\partial c_{m,n}}{\partial t} d\mathbf{x}. \end{aligned}$$

Using hypothesis (H1) – (H5) and the Lemma 3, we have

$$\begin{aligned} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 &\leq \left| b_c(\mathbf{u}_{m,n}, c_{m,n}, \frac{\partial c_{m,n}}{\partial t}) \right| + D_1 \int_{\Omega} |\Delta c_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| d\mathbf{x} \\ &\quad + c_{54} \int_{\Omega} (1 + |\nabla \psi_{m,n}|_2) |\nabla c_{m,n}|_2 \left| \frac{\partial c_{m,n}}{\partial t} \right| d\mathbf{x} + a_3 \int_{\Omega} |\Delta \psi_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| d\mathbf{x} \\ &\quad + c_{55} \int_{\Omega} (1 + |\nabla c_{m,n}|_2 + |\nabla \psi_{m,n}|_2) |\nabla \psi_{m,n}|_2 \left| \frac{\partial c_{m,n}}{\partial t} \right| d\mathbf{x}. \end{aligned}$$

and with the help of the Hölder's inequality, we have

$$\begin{aligned} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 &\leq \|\mathbf{u}_{m,n}\|_{L^4(\Omega)} \|\nabla c_{m,n}\|_{L^4(\Omega)} \left| \frac{\partial c_{m,n}}{\partial t} \right| + D_1 |\Delta c_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| \\ &\quad + c_{54} |\nabla c_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| + a_3 |\Delta \psi_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| \\ &\quad + c_{55} |\nabla \psi_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| + c_{55} \|\nabla \psi_{m,n}\|_{L^4(\Omega)}^2 \left| \frac{\partial c_{m,n}}{\partial t} \right| \\ &\quad + c_{56} \|\nabla c_{m,n}\|_{L^4(\Omega)} |\nabla \psi_{m,n}|_{L^4(\Omega)} \left| \frac{\partial c_{m,n}}{\partial t} \right|, \end{aligned}$$

and using Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 &\leq c_{57} \|\mathbf{u}_{m,n}\| \|\nabla c_{m,n}\|_{H^2(\Omega)} \|\nabla c_{m,n}\|^{\frac{1}{2}} \left| \frac{\partial c_{m,n}}{\partial t} \right| \\ &\quad + D_1 |\Delta c_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| + c_{54} |\nabla c_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| + a_3 |\Delta \psi_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| \\ &\quad + c_{55} |\nabla \psi_{m,n}| \left| \frac{\partial c_{m,n}}{\partial t} \right| + c_{58} |\nabla \psi_{m,n}| \|\nabla \psi_{m,n}\| \left| \frac{\partial c_{m,n}}{\partial t} \right| \\ &\quad + c_{59} \|c_{m,n}\|_{H^2(\Omega)} \|\psi_{m,n}\|_{H^2(\Omega)} \left| \frac{\partial c_{m,n}}{\partial t} \right|. \end{aligned}$$

Further by the Young's inequality, we have

$$\begin{aligned} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 &\leq \frac{7c_{57}^2}{2} \|\mathbf{u}_{m,n}\|^2 \|c_{m,n}\|_{H^2(\Omega)}^2 + \frac{1}{14} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 \\ &\quad + \frac{7D_1^2}{2} |\Delta c_{m,n}|^2 + \frac{1}{14} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 + \frac{7c_{54}^2}{2} |\nabla c_{m,n}|^2 + \frac{1}{14} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 \\ &\quad + \frac{7a_3^2}{2} |\Delta \psi_{m,n}|^2 + \frac{1}{14} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 + \frac{7c_{54}^2}{2} |\nabla \psi_{m,n}|^2 \\ &\quad + \frac{1}{14} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 + \frac{7c_{58}^2}{2} |\nabla \psi_{m,n}|^2 \|\nabla \psi_{m,n}\|^2 + \frac{1}{14} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 \\ &\quad + \frac{7c_{59}^2}{2} \|c_{m,n}\|_{H^2(\Omega)}^2 \|\psi_{m,n}\|_{H^2(\Omega)}^2 + \frac{1}{14} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2. \end{aligned}$$

Making use of the relations (2.57), (2.107) and (2.89), we finally arrive at

$$\frac{1}{2} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 \leq c_{60} + c_{61} \|c_{m,n}\|_{H^2(\Omega)}^2.$$

Integrating above inequality over $(0, T_f)$ and using the result (2.109), we have

$$\int_0^{T_f} \left| \frac{\partial c_{m,n}}{\partial t} \right|^2 dt \leq c_{62}.$$

Thus we have

$$\left\| \frac{\partial c_{m,n}}{\partial t} \right\|_{L^2(0, T_f; L^2(\Omega))} \leq c_{62}. \quad (2.110)$$

This shows that $\frac{\partial c_{m,n}}{\partial t}$ is uniformly bounded in $L^2(0, T_f; L^2(\Omega))$. Thus from the results (2.109) and (2.110), we can deduce that if $c_0 \in H^1(\Omega)$ then $c_{m,n}$ is uniformly bounded in \mathcal{W}_2 (defined by (2.63)). Since the embedding of $H^2(\Omega)$ in $H^1(\Omega)$ is compact, therefore we conclude that \mathcal{W}_2 is compactly embedded into $L^2(0, T_f; H^1(\Omega))$, (see e.g., [55]). Therefore we can extract from $c_{m,n}$, a subsequence also denoted by $c_{m,n}$, such that, as $m, n \rightarrow \infty$ we have

$$\begin{aligned} c_{m,n} &\rightarrow c \text{ strongly in } L^2(0, T_f; H^1(\Omega)), \\ c_{m,n} &\rightharpoonup c \text{ weakly in } L^2(0, T_f; H^2(\Omega)), \\ \frac{\partial c_{m,n}}{\partial t} &\rightharpoonup \frac{\partial c}{\partial t} \text{ weakly in } L^2(0, T_f; L^2(\Omega)), \end{aligned}$$

therefore we conclude that

$$c \in L^2(0, T_f; H^2(\Omega)) \cap H^1(0, T_f; L^2(\Omega)).$$

We can then pass easily limit, $m, n \rightarrow \infty$, to the problem (2.22)-(2.25) and verify that (\mathbf{u}, ψ, c) satisfy the problem (2.11). \square

2.6 Stability and Uniqueness

Theorem 4 *Let assumptions (H1) – (H5) be fulfilled. Let $(\mathbf{u}_{01}, \psi_{01}, c_{01}, \mathbf{B}_1)$ and $(\mathbf{u}_{02}, \psi_{02}, c_{02}, \mathbf{B}_2)$ be two functions from $(H_0^1(\Omega))^2 \times H_0^2(\Omega) \times H^1(\Omega) \times (L^2(\mathcal{Q}))^2$. If $(\mathbf{u}_1, \psi_1, c_1)$ and $(\mathbf{u}_2, \psi_2, c_2)$ are two solutions of the problem (2.11) with the given data $(\mathbf{u}_{01}, \psi_{01}, c_{01}, \mathbf{B}_1)$ and $(\mathbf{u}_{02}, \psi_{02}, c_{02}, \mathbf{B}_2)$ respectively, then we have the following estimate*

$$\begin{aligned} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{W}_1^1}^2 + \|\psi_1 - \psi_2\|_{\mathcal{W}_2^1}^2 + \|c_1 - c_2\|_{\mathcal{W}_1^1}^2 &\leq d_0 \left(\|\mathbf{u}_{01} - \mathbf{u}_{02}\|_{H^1(\Omega)}^2 \right. \\ &\quad \left. + \|\psi_{01} - \psi_{02}\|_{H^2(\Omega)}^2 + \|c_{01} - c_{02}\|_{H^1(\Omega)}^2 + \|\mathbf{B}_1 - \mathbf{B}_2\|_{L^2(\mathcal{Q})}^2 \right). \end{aligned} \quad (2.111)$$

where d_0 is constant and $\mathcal{W}_i^1 = L^\infty(0, T_f, L^2(\Omega)) \cap L^2(0, T_f, H^i(\Omega))$, $i = 1, 2$.

Proof: Let $(\mathbf{u}_1, \psi_1, c_1)$ and $(\mathbf{u}_2, \psi_2, c_2)$ be two solutions of the problem (2.11) with the given data $(\mathbf{u}_{01}, \psi_{01}, c_{01}, \mathbf{B}_1)$ and $(\mathbf{u}_{02}, \psi_{02}, c_{02}, \mathbf{B}_2)$ respectively. We denote $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\psi = \psi_1 - \psi_2$, $c = c_1 - c_2$, $\mathbf{u}_0 = \mathbf{u}_{01} - \mathbf{u}_{02}$, $\psi_0 = \psi_{01} - \psi_{02}$, $c_0 = c_{01} - c_{02}$ and $\mathbf{B} = \mathbf{B}_1 - \mathbf{B}_2$. Then the triplet (\mathbf{u}, ψ, c) is a solution of the following problem

$$\begin{aligned} \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + a_u(\mathbf{u}, \mathbf{v}) + b_u(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) - b_u(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}) \\ = (b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}_1) - b(\psi_2)((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2), \mathbf{v}) \\ + (\mathbf{A}_1(\psi_1, c_1) - \mathbf{A}_1(\psi_2, c_2), \mathbf{v}), \end{aligned} \quad (2.112)$$

$$\begin{aligned} \left(\frac{\partial \psi}{\partial t}, \phi \right) + a_\psi(\psi, \phi) + b_\psi(\mathbf{u}_1, \psi_1, \phi) - b_\psi(\mathbf{u}_2, \psi_2, \phi) \\ = - \int_{\Omega} (A_2(\psi_1, c_1) - A_2(\psi_2, c_2)) \phi \, d\mathbf{x}, \end{aligned} \quad (2.113)$$

$$\begin{aligned} \left(\frac{\partial c}{\partial t}, z \right) + b_c(\mathbf{u}_1, c_1, z) - b_c(\mathbf{u}_2, c_2, z) + \int_{\Omega} D(\psi_1) \nabla c_1 \cdot \nabla z \, d\mathbf{x} \\ - \int_{\Omega} D(\psi_2) \nabla c_2 \cdot \nabla z \, d\mathbf{x} + \int_{\Omega} A_3(\psi_1, c_1) \nabla \psi_1 \cdot \nabla z \, d\mathbf{x} \\ - \int_{\Omega} A_3(\psi_2, c_2) \nabla \psi_2 \cdot \nabla z \, d\mathbf{x} = 0, \end{aligned} \quad (2.114)$$

$$(\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0). \quad \forall (\mathbf{v}, \phi, z) \in \mathbf{V} \quad (2.115)$$

Consider now the term

$$b_u(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) - b_u(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}) = \int_{\Omega} (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1) \cdot \mathbf{v} \, d\mathbf{x} - \int_{\Omega} (\mathbf{u}_2 \cdot \nabla \mathbf{u}_2) \cdot \mathbf{v} \, d\mathbf{x},$$

adding and subtracting the term $\int_{\Omega} (\mathbf{u}_2 \cdot \nabla \mathbf{u}_1) \cdot \mathbf{v} \, d\mathbf{x}$, we have

$$b_u(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) - b_u(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}) = \int_{\Omega} (\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla \mathbf{u}_1 \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mathbf{u}_2 \cdot \nabla (\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{v} \, d\mathbf{x},$$

thus we have

$$b_u(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) - b_u(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}) = b_u(\mathbf{u}, \mathbf{u}_1, \mathbf{v}) + b_u(\mathbf{u}_2, \mathbf{u}, \mathbf{v}). \quad (2.116)$$

Similarly, we can derive

$$\left. \begin{aligned} b_\psi(\mathbf{u}_1, \psi_1, \phi) - b_\psi(\mathbf{u}_2, \psi_2, \phi) &= b_\psi(\mathbf{u}, \psi_1, \phi) + b_\psi(\mathbf{u}_2, \psi, \phi), \\ b_c(\mathbf{u}_1, c_1, z) - b_c(\mathbf{u}_2, c_2, z) &= b_c(\mathbf{u}, c_1, z) + b_c(\mathbf{u}_2, c, z). \end{aligned} \right\} \quad (2.117)$$

Consider the term

$$b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}_1) - b(\psi_2)((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2),$$

adding and subtracting the terms $b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}_2)$, $b(\psi_1)((\mathbf{u}_2 \times \mathbf{B}_1) \times \mathbf{B}_2)$ and $b(\psi_1)((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2)$ in the above expression and simplifying, we have

$$\begin{aligned} & b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}_1) - b(\psi_2)((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2) \\ &= b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}) + b(\psi_1)((\mathbf{u} \times \mathbf{B}_1) \times \mathbf{B}_2) \\ &+ b(\psi_1)((\mathbf{u}_2 \times \mathbf{B}) \times \mathbf{B}_2) + (b(\psi_1) - b(\psi_2))((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2), \end{aligned}$$

the above equation can further be written as

$$\begin{aligned} & b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}_1) - b(\psi_2)((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2) \\ &= b(\psi_1)\{((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}) + ((\mathbf{u} \times \mathbf{B}_1) \times \mathbf{B}_2) \\ &+ ((\mathbf{u}_2 \times \mathbf{B}) \times \mathbf{B}_2)\} + (b(\psi_1) - b(\psi_2))((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2). \end{aligned} \quad (2.118)$$

Making use of equations (2.116)-(2.118) in the equations (2.112)-(2.114) and adding and subtracting the terms $\int_{\Omega} D(\psi_2) \nabla c_1 \cdot \nabla z d\mathbf{x}$ and $\int_{\Omega} A_3(\psi_2, c_2) \nabla \psi_1 \cdot \nabla z d\mathbf{x}$ in equation (2.114), we obtain

$$\begin{aligned} & \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + a_u(\mathbf{u}, \mathbf{v}) + b_u(\mathbf{u}, \mathbf{u}_1, \mathbf{v}) + b_u(\mathbf{u}_2, \mathbf{u}, \mathbf{v}) \\ &= \left(\mathbf{A}_1(\psi_1, c_1) - \mathbf{A}_1(\psi_2, c_2), \mathbf{v} \right) + \left(b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}), \mathbf{v} \right) \\ &+ \left(b(\psi_1)((\mathbf{u} \times \mathbf{B}_1) \times \mathbf{B}_2), \mathbf{v} \right) + \left(b(\psi_1)(\mathbf{u}_2 \times \mathbf{B}) \times \mathbf{B}_2, \mathbf{v} \right) \\ &+ \left((b(\psi_1) - b(\psi_2))((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2), \mathbf{v} \right), \end{aligned} \quad (2.119)$$

$$\begin{aligned} & \left(\frac{\partial \psi}{\partial t}, \phi \right) + a_{\psi}(\psi, \phi) + b_{\psi}(\mathbf{u}, \psi_1, \phi) + b_{\psi}(\mathbf{u}_2, \psi, \phi) \\ &= - \int_{\Omega} (A_2(\psi_1, c_1) - A_2(\psi_2, c_2)) \phi d\mathbf{x}, \end{aligned} \quad (2.120)$$

$$\begin{aligned} & \left(\frac{\partial c}{\partial t}, z \right) + b_c(\mathbf{u}, c_1, z) + b_c(\mathbf{u}_2, c, z) + \int_{\Omega} (D(\psi_1) - D(\psi_2)) \nabla c_1 \cdot \nabla z d\mathbf{x} \\ &+ \int_{\Omega} D(\psi_2) \nabla c \cdot \nabla z d\mathbf{x} + \int_{\Omega} (A_3(\psi_1, c_1) - A_3(\psi_2, c_2)) \nabla \psi_1 \cdot \nabla z d\mathbf{x} \\ &+ \int_{\Omega} A_3(\psi_2, c_2) \nabla \psi \cdot \nabla z d\mathbf{x} = 0. \end{aligned} \quad (2.121)$$

Now setting $(\mathbf{v}, \phi, z) = (\mathbf{u}, \psi, c)$ in the equations (2.119)-(2.121), we have

$$\begin{aligned} & \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{u} \right) + a_u(\mathbf{u}, \mathbf{u}) + b_u(\mathbf{u}, \mathbf{u}_1, \mathbf{u}) + b_u(\mathbf{u}_2, \mathbf{u}, \mathbf{u}) \\ &= \left(\mathbf{A}_1(\psi_1, c_1) - \mathbf{A}_1(\psi_2, c_2), \mathbf{u} \right) + \left(b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}), \mathbf{u} \right) \\ & \quad + \left(b(\psi_1)(\mathbf{u} \times \mathbf{B}_1) \times \mathbf{B}_2, \mathbf{u} \right) + \left(b(\psi_1)(\mathbf{u}_2 \times \mathbf{B}) \times \mathbf{B}_2, \mathbf{u} \right) \\ & \quad + \left((b(\psi_1) - b(\psi_2))((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2), \mathbf{u} \right), \end{aligned}$$

$$\begin{aligned} & \left(\frac{\partial \psi}{\partial t}, \psi \right) + a_\psi(\psi, \psi) + b_\psi(\mathbf{u}, \psi_1, \psi) + b_\psi(\mathbf{u}_2, \psi, \psi) \\ &= - \int_{\Omega} (A_2(\psi_1, c_1) - A_2(\psi_2, c_2)) \psi \, d\mathbf{x}, \end{aligned}$$

$$\begin{aligned} & \left(\frac{\partial c}{\partial t}, c \right) + b_c(\mathbf{u}, c_1, c) + b_c(\mathbf{u}_2, c, c) + \int_{\Omega} (D(\psi_1) - D(\psi_2)) \nabla c_1 \cdot \nabla c \, d\mathbf{x} \\ & \quad + \int_{\Omega} D(\psi_2) \nabla c \cdot \nabla c \, d\mathbf{x} + \int_{\Omega} (A_3(\psi_1, c_1) - A_3(\psi_2, c_2)) \nabla \psi_1 \cdot \nabla c \, d\mathbf{x} \\ & \quad + \int_{\Omega} A_3(\psi_2, c_2) \nabla \psi \cdot \nabla c \, d\mathbf{x} = 0, \end{aligned}$$

since $b_u(\mathbf{u}_2, \mathbf{u}, \mathbf{u}) = b_\psi(\mathbf{u}_2, \psi, \psi) = b_c(\mathbf{u}_2, c, c) = 0$ and $b_\psi(\mathbf{u}, \psi_1, \psi) = -b_\psi(\mathbf{u}, \psi, \psi_1)$, the above equations takes the form

$$\begin{aligned} & \frac{\rho_0}{2} \frac{d}{dt} |\mathbf{u}|^2 + \mu \int_{\Omega} |\nabla \mathbf{u}|_2^2 \, d\mathbf{x} + b_u(\mathbf{u}, \mathbf{u}_1, \mathbf{u}) = \left(\mathbf{A}_1(\psi_1, c_1) - \mathbf{A}_1(\psi_2, c_2), \mathbf{u} \right) \\ & \quad + \left(b(\psi_1)((\mathbf{u}_1 \times \mathbf{B}_1) \times \mathbf{B}), \mathbf{u} \right) + \left(b(\psi_1)(\mathbf{u} \times \mathbf{B}_1) \times \mathbf{B}_2, \mathbf{u} \right) \\ & \quad + \left(b(\psi_1)(\mathbf{u}_2 \times \mathbf{B}) \times \mathbf{B}_2, \mathbf{u} \right) + \left((b(\psi_1) - b(\psi_2))((\mathbf{u}_2 \times \mathbf{B}_2) \times \mathbf{B}_2), \mathbf{u} \right), \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |\psi|^2 + \epsilon_1 \int_{\Omega} |\nabla \psi|_2^2 \, d\mathbf{x} - b_\psi(\mathbf{u}, \psi, \psi_1) \\ &= - \int_{\Omega} (A_2(\psi_1, c_1) - A_2(\psi_2, c_2)) \psi \, d\mathbf{x}, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |c|^2 + \int_{\Omega} D(\psi_2) |\nabla c|_2^2 \, d\mathbf{x} + b_c(\mathbf{u}, c_1, c) = - \int_{\Omega} A_3(\psi_2, c_2) \nabla \psi \cdot \nabla c \, d\mathbf{x} \\ & \quad - \int_{\Omega} (D(\psi_1) - D(\psi_2)) \nabla c_1 \cdot \nabla c \, d\mathbf{x} \\ & \quad - \int_{\Omega} (A_3(\psi_1, c_1) - A_3(\psi_2, c_2)) \nabla \psi_1 \cdot \nabla c \, d\mathbf{x}, \end{aligned}$$

and using hypothesis (H1) – (H5), the above equations takes the form

$$\begin{aligned}
\frac{\rho_0}{2} \frac{d}{dt} |\mathbf{u}|^2 + \mu |\nabla \mathbf{u}|^2 &\leq \rho_0 \int_{\Omega} |\mathbf{u}|_2 |\nabla \mathbf{u}_1|_2 |\mathbf{u}|_2 d\mathbf{x} + d_1 \int_{\Omega} (|\psi| + |c|) |\mathbf{u}|_2 d\mathbf{x} \\
&\quad + d_2 \int_{\Omega} |\mathbf{u}_1|_2 |\mathbf{u}|_2 |\mathbf{B}|_2 d\mathbf{x} + d_3 \int_{\Omega} |\mathbf{u}|_2^2 d\mathbf{x} \\
&\quad + d_4 \int_{\Omega} |\mathbf{u}_2|_2 |\mathbf{u}|_2 |\mathbf{B}|_2 d\mathbf{x} + d_5 \int_{\Omega} |\psi| |\mathbf{u}_2|_2 |\mathbf{u}|_2 d\mathbf{x} \\
\frac{1}{2} \frac{d}{dt} |\psi|^2 + \epsilon_1 |\nabla \psi|^2 &\leq d_6 \int_{\Omega} (|\psi| + |c|) |\psi| d\mathbf{x} + \int_{\Omega} |\mathbf{u}|_2 |\nabla \psi|_2 |\psi|_1 d\mathbf{x}, \\
\frac{1}{2} \frac{d}{dt} |c|^2 + D_0 |\nabla c|^2 &\leq \int_{\Omega} |\mathbf{u}|_2 |\nabla c_1|_2 |c| d\mathbf{x} + a_3 \int_{\Omega} |\nabla \psi|_2 |\nabla c|_2 d\mathbf{x} \\
&\quad + d_7 \int_{\Omega} |\psi| |\nabla c_1|_2 |\nabla c|_2 d\mathbf{x} + d_8 \int_{\Omega} (|\psi| + |c|) |\nabla \psi|_2 |\nabla c|_2 d\mathbf{x}.
\end{aligned}$$

Let $\nu = \mu/\rho_0$. As $\psi_1 \in L^\infty(0, T_f, H^2(\Omega)) \subset L^\infty(\mathcal{Q})$ and using Hölder's inequality, we have

$$\begin{aligned}
\frac{d}{dt} |\mathbf{u}|^2 + 2\nu |\nabla \mathbf{u}|^2 &\leq 2 \|\mathbf{u}\|_{L^4(\Omega)}^2 |\nabla \mathbf{u}_1| + d_9 \left(|\mathbf{u}| |\psi| + |\mathbf{u}| |c| \right) \\
&\quad + d_{10} \|\mathbf{u}_1\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{B}| + d_{11} \|\mathbf{u}_2\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{B}| \\
&\quad + d_{12} |\mathbf{u}|^2 + d_{13} \|\psi\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{u}_2|, \tag{2.122}
\end{aligned}$$

$$\frac{d}{dt} |\psi|^2 + 2\epsilon_1 |\nabla \psi|^2 \leq d_{14} (|\psi|^2 + |c| |\psi|) + 2 \|\psi_1\|_{L^\infty(\mathcal{Q})} |\mathbf{u}| |\nabla \psi|, \tag{2.123}$$

$$\begin{aligned}
\frac{d}{dt} |c|^2 + 2D_0 |\nabla c|^2 &\leq 2 \|\mathbf{u}\|_{L^4(\Omega)} \|c\|_{L^4(\Omega)} |\nabla c_1| + d_{15} |\nabla \psi| |\nabla c| \\
&\quad + d_{16} \int_{\Omega} |\psi| |\nabla c_1|_2 |\nabla c|_2 d\mathbf{x} + d_{17} \int_{\Omega} (|\psi| + |c|) |\nabla \psi|_2 |\nabla c|_2 d\mathbf{x}. \tag{2.124}
\end{aligned}$$

Since $\mathbf{u}_1 \in L^\infty(0, T_f; V)$ and using Gagliardo-Nirenberg's inequality, we have

$$2 \|\mathbf{u}\|_{L^4(\Omega)}^2 |\nabla \mathbf{u}_1| \leq d_{18} |\mathbf{u}| |\nabla \mathbf{u}|,$$

Further with the help of Young's inequality, we obtain

$$2 \|\mathbf{u}\|_{L^4(\Omega)}^2 |\nabla \mathbf{u}_1| \leq \frac{d_{18}^2}{\nu} |\mathbf{u}|^2 + \frac{\nu}{4} |\nabla \mathbf{u}|^2 \tag{2.125}$$

As $\mathbf{u}_2 \in L^\infty(0, T_f; V)$, and using Gagliardo-Nirenberg's inequality, we get

$$d_{13} \|\psi\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{u}_2| \leq d_{19} \|\psi\| |\mathbf{u}|^{1/2} |\nabla \mathbf{u}|^{1/2}.$$

Again by using Young's inequality

$$d_{13} \|\psi\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{u}_2| \leq \frac{d_{19}^2}{2} \|\psi\|^2 + \frac{1}{4\nu} |\mathbf{u}|^2 + \frac{\nu}{4} |\nabla \mathbf{u}|^2. \quad (2.126)$$

As $\mathbf{u}_1 \in L^\infty(0, T_f; V)$ and applying Gagliardo-Nirenberg's inequality, we have

$$d_{10} \|\mathbf{u}_1\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{B}| \leq d_{20} |\mathbf{u}|^{1/2} |\nabla \mathbf{u}|^{1/2} |\mathbf{B}|,$$

and then (using Young's inequality), we arrive at

$$d_{10} \|\mathbf{u}_1\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{B}| \leq \frac{d_{20}^4}{4\nu} |\mathbf{u}|^2 + \frac{\nu}{4} |\nabla \mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2. \quad (2.127)$$

Similarly we can derive that

$$d_{11} \|\mathbf{u}_2\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} |\mathbf{B}| \leq \frac{d_{21}^4}{4\nu} |\mathbf{u}|^2 + \frac{\nu}{4} |\nabla \mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2. \quad (2.128)$$

Also as $c_1 \in L^\infty(0, T_f; H^1(\Omega))$, thus we have

$$2 \|\mathbf{u}\|_{L^4(\Omega)} \|c\|_{L^4(\Omega)} |\nabla c_1| \leq d_{22} \|\mathbf{u}\|_{L^4(\Omega)} \|c\|_{L^4(\Omega)}.$$

According to the Sobolev injection ($H^1(\Omega) \subset L^4(\Omega)$), and as $\|\mathbf{u}\|_V \leq d_{23} |\nabla \mathbf{u}|$ we have

$$2 \|\mathbf{u}\|_{L^4(\Omega)} \|c\|_{L^4(\Omega)} |\nabla c_1| \leq d_{24} |\nabla \mathbf{u}| \|c\|_{H^1(\Omega)},$$

the above inequality can further be written as

$$2 \|\mathbf{u}\|_{L^4(\Omega)} \|c\|_{L^4(\Omega)} |\nabla c_1| \leq d_{24} |\nabla \mathbf{u}| (|c|^2 + |\nabla c|^2)^{1/2}.$$

Applying the Young's inequality, we have

$$2 \|\mathbf{u}\|_{L^4(\Omega)} \|c\|_{L^4(\Omega)} |\nabla c_1| \leq d_{25} (|c|^2 + |\nabla \mathbf{u}|^2) + \frac{D_0}{4} |\nabla c|^2. \quad (2.129)$$

Since $\psi \in H^2(\Omega) \subset L^\infty(\Omega)$, therefore we obtain

$$d_{16} \int_{\Omega} |\psi| |\nabla c_1|_2 |\nabla c|_2 d\mathbf{x} \leq d_{16} \|\psi\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla c_1|_2 |\nabla c|_2 d\mathbf{x}$$

Using Sobolev injection ($H^2(\Omega) \subset L^\infty(\Omega)$) and Hölder's inequality, we arrive at

$$d_{16} \int_{\Omega} |\psi| |\nabla c_1|_2 |\nabla c|_2 d\mathbf{x} \leq d_{26} \|\psi\|_{H^2(\Omega)} |\nabla c_1| |\nabla c|$$

Since $c_1 \in L^\infty(0, T_f; H^1(\Omega))$ and according to the elliptic estimate, we can write

$$d_{16} \int_{\Omega} |\psi| |\nabla c_1|_2 |\nabla c|_2 d\mathbf{x} \leq d_{27} (|\Delta \psi| + |\psi|) |\nabla c|,$$

and with the help of Young's inequality, the above inequality becomes

$$d_{16} \int_{\Omega} |\psi| |\nabla c|_2 |\nabla c|_2 d\mathbf{x} \leq d_{28} (|\Delta \psi|^2 + |\psi|^2) + \frac{D_0}{4} |\nabla c|^2 \quad (2.130)$$

Now using Hölder's inequality, we have

$$\begin{aligned} d_{17} \int_{\Omega} (|\psi| + |c|) |\nabla \psi_1|_2 |\nabla c|_2 d\mathbf{x} &\leq \\ d_{17} \left(\|\psi\|_{L^4(\Omega)} + \|c\|_{L^4(\Omega)} \right) \|\nabla \psi_1\|_{L^4(\Omega)} |\nabla c|, \end{aligned}$$

since $\psi_1 \in L^\infty(0, T_f; H^2(\Omega))$ and using Gagliardo-Nirenberg's inequality, we finally get

$$d_{17} \int_{\Omega} (|\psi| + |c|) |\nabla \psi_1|_2 |\nabla c|_2 d\mathbf{x} \leq d_{29} \left(\|\psi\| + |c|^{\frac{1}{2}} \|c\|^{\frac{1}{2}} \right) |\nabla c|.$$

The above inequality can further be written as

$$d_{17} \int_{\Omega} (|\psi| + |c|) |\nabla \psi_1|_2 |\nabla c|_2 d\mathbf{x} \leq d_{30} \left(\|\psi\| + |c|^{\frac{1}{2}} \left(|c|^{\frac{1}{2}} + |\nabla c|^{\frac{1}{2}} \right) \right) |\nabla c|,$$

or

$$\begin{aligned} d_{17} \int_{\Omega} (|\psi| + |c|) |\nabla \psi_1|_2 |\nabla c|_2 d\mathbf{x} &\leq \\ d_{30} \left(\|\psi\| |\nabla c| + |c| |\nabla c| + |c|^{\frac{1}{2}} |\nabla c|^{\frac{3}{2}} \right), \end{aligned}$$

and applying the Young's formula, we have

$$d_{17} \int_{\Omega} (|\psi| + |c|) |\nabla \psi_1|_2 |\nabla c|_2 d\mathbf{x} \leq d_{31} (\|\psi\|^2 + |c|^2) + \frac{D_0}{4} |\nabla c|^2. \quad (2.131)$$

Making use of the inequalities (2.125)-(2.131) in the the relations (2.122)-(2.124), we have

$$\begin{aligned} \frac{d}{dt} |\mathbf{u}|^2 + \nu |\nabla \mathbf{u}|^2 &\leq \frac{d_{18}^2}{\nu} |\mathbf{u}|^2 + d_9 (|\psi| |\mathbf{u}| + |c| |\mathbf{u}|) + \frac{d_{20}^4}{4\nu} |\mathbf{u}|^2 + |\mathbf{B}|^2 \\ &\quad + \frac{d_{21}^4}{4\nu} |\mathbf{u}|^2 + d_{12} |\mathbf{u}|^2 + \frac{d_{19}^2}{2} \|\psi\|^2 + \frac{1}{4\nu} |\mathbf{u}|^2 \end{aligned}$$

$$\frac{d}{dt} |\psi|^2 + 2\epsilon_1 |\nabla \psi|^2 \leq d_{14} (|\psi|^2 + |c| |\psi|) + d_{32} |\mathbf{u}| |\nabla \psi|,$$

$$\begin{aligned} \frac{d}{dt} |c|^2 + 2D_0 |\nabla c|^2 &\leq \frac{3D_0}{4} |\nabla c|^2 + d_{25} (|c|^2 + |\nabla \mathbf{u}|^2) + d_{15} |\nabla \psi| |\nabla c| \\ &\quad + d_{28} (|\Delta \psi|^2 + |\psi|^2) + d_{31} (\|\psi\|^2 + |c|^2). \end{aligned}$$

By the Young's formula, the above inequalities takes the following form

$$\frac{d}{dt} |\mathbf{u}|^2 + \nu |\nabla \mathbf{u}|^2 \leq d_{33} (|\mathbf{u}|^2 + |\psi|^2 + |c|^2) + |\mathbf{B}|^2 + \frac{d_{19}^2}{2} \|\psi\|^2,$$

$$\frac{d}{dt} |\psi|^2 + 2\epsilon_1 |\nabla \psi|^2 \leq d_{34} (|\mathbf{u}|^2 + |\psi|^2 + |c|^2) + \epsilon_1 |\nabla \psi|^2,$$

$$\begin{aligned} \frac{d}{dt} |c|^2 + D_0 |\nabla c|^2 &\leq d_{25} (|c|^2 + |\nabla \mathbf{u}|^2) + \frac{d_{15}^2}{D_0} |\nabla \psi|^2 \\ &\quad + d_{28} (|\Delta \psi|^2 + |\psi|^2) + d_{31} (\|\psi\|^2 + |c|^2). \end{aligned}$$

The above inequalities can further be written as

$$\frac{d}{dt} |\mathbf{u}|^2 + \nu |\nabla \mathbf{u}|^2 \leq d_{35} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + |\mathbf{B}|^2, \quad (2.132)$$

$$\frac{d}{dt} |\psi|^2 + \epsilon_1 |\nabla \psi|^2 \leq d_{36} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2), \quad (2.133)$$

$$\frac{d}{dt} |c|^2 + D_0 |\nabla c|^2 \leq d_{37} (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2 + |c|^2 + \|\psi\|^2). \quad (2.134)$$

Now we shall estimate $|\Delta \psi|$. For this setting $\phi = -\Delta \psi$ in equation (2.113), we obtain

$$\begin{aligned} - \left(\frac{\partial \psi}{\partial t}, \Delta \psi \right) - \epsilon_1 \int_{\Omega} \nabla \psi \cdot \nabla (\Delta \psi) d\mathbf{x} &= b_{\psi}(\mathbf{u}_1, \psi_1, \Delta \psi) - b_{\psi}(\mathbf{u}_2, \psi_2, \Delta \psi) \\ &\quad + \int_{\Omega} (A_2(\psi_1, c_1) - A_2(\psi_2, c_2)) \Delta \psi d\mathbf{x}, \end{aligned}$$

Making use of equation (2.117) and Green's formula, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\nabla \psi|^2 + \epsilon_1 |\Delta \psi|^2 &= b_{\psi}(\mathbf{u}, \psi_1, \Delta \psi) - b_{\psi}(\mathbf{u}_2, \psi, \Delta \psi) \\ &\quad + \int_{\Omega} (A_2(\psi_1, c_1) - A_2(\psi_2, c_2)) \Delta \psi d\mathbf{x}, \end{aligned}$$

and then using hypothesis (H1) – (H5), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\nabla \psi|^2 + \epsilon_1 |\Delta \psi|^2 &\leq \int_{\Omega} |\mathbf{u}|_2 |\nabla \psi_1|_2 |\Delta \psi| d\mathbf{x} \\ &\quad + \int_{\Omega} |\mathbf{u}_2|_2 |\nabla \psi|_2 |\Delta \psi| d\mathbf{x} + \int_{\Omega} (|\psi| + |c|) |\Delta \psi| d\mathbf{x}, \end{aligned}$$

with the help of the Hölder's inequality, we have

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + 2\epsilon_1 |\Delta \psi|^2 &\leq 2 \|\mathbf{u}\|_{L^4(\Omega)} \|\nabla \psi_1\|_{L^4(\Omega)} |\Delta \psi| \\ &\quad + 2 \|\mathbf{u}_2\|_{L^4(\Omega)} \|\nabla \psi\|_{L^4(\Omega)} |\Delta \psi| + 2 (|\psi| + |c|) |\Delta \psi|. \end{aligned}$$

According to the Lemma 1, we arrive at

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + 2\epsilon_1 |\Delta \psi|^2 &\leq d_{38} |\mathbf{u}|^{1/2} |\nabla \mathbf{u}|^{1/2} \|\nabla \psi_1\| |\Delta \psi| \\ &\quad + d_{39} |\nabla \mathbf{u}_2| |\nabla \psi|^{1/2} \|\nabla \psi\|^{1/2} |\Delta \psi| + 2(|\psi| + |c|) |\Delta \psi|. \end{aligned}$$

Since $\psi_1 \in L^\infty(0, T_f, H^2(\Omega))$ and $\mathbf{u}_2 \in L^\infty(0, T_f, V)$, therefore the above inequality takes the form

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + 2\epsilon_1 |\Delta \psi|^2 &\leq d_{40} |\mathbf{u}|^{1/2} |\nabla \mathbf{u}|^{1/2} |\Delta \psi| + d_{41} |\nabla \psi|^{1/2} \|\nabla \psi\|^{1/2} |\Delta \psi| \\ &\quad + 2(|\psi| + |c|) |\Delta \psi|, \end{aligned}$$

and using Young's formula, we arrive at

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + 2\epsilon_1 |\Delta \psi|^2 &\leq \frac{d_{40}^2}{\epsilon_1} |\mathbf{u}| |\nabla \mathbf{u}| + \frac{\epsilon_1}{4} |\Delta \psi|^2 + \frac{d_{41}^2}{\epsilon_1} |\nabla \psi| \|\nabla \psi\| \\ &\quad + \frac{\epsilon_1}{4} |\Delta \psi|^2 + \frac{4}{\epsilon_1} (|\psi|^2 + |c|^2) + \frac{\epsilon_1}{4} |\Delta \psi|^2. \end{aligned}$$

The above inequality can further be written as

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + 2\epsilon_1 |\Delta \psi|^2 &\leq \frac{d_{40}^2}{\epsilon_1} |\mathbf{u}| |\nabla \mathbf{u}| + d_{42} \|\psi\| \|\psi\|_{H^2(\Omega)} \\ &\quad + \frac{4}{\epsilon_1} (|\psi|^2 + |c|^2) + \frac{3\epsilon_1}{4} |\Delta \psi|^2, \end{aligned}$$

and using elliptic estimate, we can write

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + 2\epsilon_1 |\Delta \psi|^2 &\leq \frac{d_{40}^2}{\epsilon_1} |\mathbf{u}| |\nabla \mathbf{u}| + d_{43} \|\psi\| (|\psi| + |\Delta \psi|) \\ &\quad + \frac{4}{\epsilon_1} (|\psi|^2 + |c|^2) + \frac{3\epsilon_1}{4} |\Delta \psi|^2, \end{aligned}$$

again by applying the Young's formula, we have

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + 2\epsilon_1 |\Delta \psi|^2 &\leq \frac{d_{40}^4}{2\nu\epsilon_1^2} |\mathbf{u}|^2 + \frac{\nu}{2} |\nabla \mathbf{u}|^2 + \frac{d_{47}^2}{2} \|\psi\|^2 + \frac{1}{2} |\psi|^2 \\ &\quad + \frac{d_{43}^2}{\epsilon_1} \|\psi\|^2 + \frac{\epsilon_1}{4} |\Delta \psi|^2 + \frac{4}{\epsilon_1} (|\psi|^2 + |c|^2) + \frac{3\epsilon_1}{4} |\Delta \psi|^2. \end{aligned}$$

Simplifying above inequality, we arrive at

$$\begin{aligned} \frac{d}{dt} |\nabla \psi|^2 + \epsilon_1 |\Delta \psi|^2 &\leq \frac{d_{40}^4}{2\nu\epsilon_1^2} |\mathbf{u}|^2 + \frac{\nu}{2} |\nabla \mathbf{u}|^2 \\ &\quad + d_{44} \|\psi\|^2 + \frac{4}{\epsilon_1} |c|^2. \end{aligned}$$

Consequently

$$\frac{d}{dt} |\nabla \psi|^2 + \epsilon_1 |\Delta \psi|^2 \leq d_{45} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + \frac{\nu}{2} |\nabla \mathbf{u}|^2. \quad (2.135)$$

Now adding the relations (2.132), (2.133) and (2.135), we arrive at

$$\begin{aligned} \frac{d}{dt} (|\mathbf{u}|^2 + \|\psi\|^2) + \frac{\nu}{2} |\nabla \mathbf{u}|^2 + \epsilon_1 |\nabla \psi|^2 + \epsilon_1 |\Delta \psi|^2 \\ \leq d_{46} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + |\mathbf{B}|^2. \end{aligned}$$

Let $\delta_1 = \min(\nu/2, \epsilon_1)$, then the above inequality takes the form

$$\begin{aligned} \frac{d}{dt} (|\mathbf{u}|^2 + \|\psi\|^2) + \delta_1 (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2 + |\nabla \psi|^2) \\ \leq d_{46} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + |\mathbf{B}|^2. \end{aligned} \quad (2.136)$$

Multiplying inequality (2.134) by $\delta_2 > 0$ and then adding the resulting inequality with (2.136), we finally arrive at

$$\begin{aligned} \frac{d}{dt} (|\mathbf{u}|^2 + \|\psi\|^2 + \delta_2 |c|^2) + \delta_1 (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2 + |\nabla \psi|^2) + \delta_2 D_0 |\nabla c|^2 \\ \leq d_{46} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + |\mathbf{B}|^2 + \delta_2 d_{37} (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2 + |c|^2 + \|\psi\|^2) \end{aligned}$$

Choosing $\delta_2 = \delta_1/2d_{37}$, the above inequality takes the form

$$\begin{aligned} \frac{d}{dt} (|\mathbf{u}|^2 + \|\psi\|^2 + \delta_2 |c|^2) + \frac{\delta_1}{2} (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2) + \delta_1 |\nabla \psi|^2 + \delta_2 D_0 |\nabla c|^2 \\ \leq d_{46} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + |\mathbf{B}|^2 + \frac{\delta_1}{2} (|c|^2 + \|\psi\|^2), \end{aligned}$$

and simplifying above inequality we arrive at

$$\begin{aligned} \frac{d}{dt} (|\mathbf{u}|^2 + \|\psi\|^2 + \delta_2 |c|^2) + \frac{\delta_1}{2} (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2) + \delta_1 |\nabla \psi|^2 + \delta_2 D_0 |\nabla c|^2 \\ \leq d_{47} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + |\mathbf{B}|^2. \end{aligned} \quad (2.137)$$

From the above inequality (2.137), we can deduce that

$$\frac{d}{dt} (|\mathbf{u}|^2 + \|\psi\|^2 + \delta_2 |c|^2) \leq d_{47} (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) + |\mathbf{B}|^2,$$

and using Gronwall's lemma and the relation (2.115), we have

$$|\mathbf{u}(t)|^2 + \|\psi(t)\|^2 + |c(t)|^2 \leq d_{48} (|\mathbf{u}_0|^2 + \|\psi_0\|^2 + |c_0|^2 + |\mathbf{B}|^2), \quad \forall t \in (0, T_f)$$

therefore we have

$$\begin{aligned} \|\mathbf{u}(t)\|_{L^\infty(0,T_f,L^2(\Omega))} + \|\psi(t)\|_{L^\infty(0,T_f,H^1(\Omega))} + \|c(t)\|_{L^\infty(0,T_f,L^2(\Omega))} \\ \leq d_{48} \left(|\mathbf{u}_0|^2 + \|\psi_0\|^2 + |c_0|^2 + |\mathbf{B}|^2 \right). \end{aligned} \quad (2.138)$$

Now integrating the inequality (2.137) over $(0, t)$ for all $t \in (0, T_f)$, we have

$$\begin{aligned} (|\mathbf{u}(t)|^2 + \|\psi(t)\|^2 + \delta_2 |c(t)|^2) + \frac{\delta_1}{2} \int_0^t (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2) ds + \delta_1 \int_0^t |\nabla \psi|^2 ds \\ + \delta_2 D_0 \int_0^t |\nabla c|^2 ds \leq d_{47} \int_0^t (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) ds + \int_0^t |\mathbf{B}|^2 ds \\ + (|\mathbf{u}(0)|^2 + \|\psi(0)\|^2 + \delta_2 |c(0)|^2). \end{aligned}$$

From the above inequality, we can deduce that

$$\begin{aligned} \frac{\delta_1}{2} \int_0^t (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2) ds + \delta_1 \int_0^t |\nabla \psi|^2 ds + \delta_2 D_0 \int_0^t |\nabla c|^2 ds \\ \leq d_{47} \int_0^t (|\mathbf{u}|^2 + \|\psi\|^2 + |c|^2) ds + \int_0^t |\mathbf{B}|^2 ds \\ + (|\mathbf{u}(0)|^2 + \|\psi(0)\|^2 + \delta_2 |c(0)|^2), \end{aligned}$$

and according to the relation (2.138), we arrive at

$$\begin{aligned} \frac{\delta_1}{2} \int_0^t (|\nabla \mathbf{u}|^2 + |\Delta \psi|^2) ds + \delta_1 \int_0^t |\nabla \psi|^2 ds + \delta_2 D_0 \int_0^t |\nabla c|^2 ds \\ \leq d_{48} \left((|\mathbf{u}(0)|^2 + \|\psi(0)\|^2 + \delta_2 |c(0)|^2) + \int_0^t |\mathbf{B}|^2 ds \right). \end{aligned} \quad (2.139)$$

From the relations (2.138) and (2.139), we can easily have

$$\begin{aligned} \|\mathbf{u}(t)\|_{\mathcal{W}_1^1}^2 + \|\psi(t)\|_{\mathcal{W}_2^1}^2 + \|c(t)\|_{\mathcal{W}_1^1}^2 \leq d_0 \left(\|\mathbf{u}_0\|_{H^1(\Omega)}^2 \right. \\ \left. + \|\psi_0\|_{H^2(\Omega)}^2 + \|c_0\|_{H^1(\Omega)}^2 + \|\mathbf{B}\|_{L^2(\mathcal{Q})}^2 \right). \end{aligned} \quad (2.140)$$

As $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\psi = \psi_1 - \psi_2$, $c = c_1 - c_2$, $\mathbf{u}_0 = \mathbf{u}_{01} - \mathbf{u}_{02}$, $\psi_0 = \psi_{01} - \psi_{02}$, $c_0 = c_{01} - c_{02}$ and $\mathbf{B} = \mathbf{B}_1 - \mathbf{B}_2$, therefore we can write

$$\begin{aligned} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{W}_1^1}^2 + \|\psi_1 - \psi_2\|_{\mathcal{W}_2^1}^2 + \|c_1 - c_2\|_{\mathcal{W}_1^1}^2 \leq d_0 \left(\|\mathbf{u}_{01} - \mathbf{u}_{02}\|_{H^1(\Omega)}^2 \right. \\ \left. + \|\psi_{01} - \psi_{02}\|_{H^2(\Omega)}^2 + \|c_{01} - c_{02}\|_{H^1(\Omega)}^2 + \|\mathbf{B}_1 - \mathbf{B}_2\|_{L^2(\mathcal{Q})}^2 \right). \end{aligned} \quad (2.141)$$

which is the required result. \square

Corollary 1 *The solution of the problem (2.11) is unique.*

Proof: If we assume in the Theorem 4 that the given data is same *i.e.*, $\mathbf{u}_{01} = \mathbf{u}_{02}$, $\psi_{01} = \psi_{02}$, $c_{01} = c_{02}$ and $\mathbf{B}_1 = \mathbf{B}_2$. Then we obtain $(\mathbf{u}, \psi, c) = (\mathbf{0}, 0, 0)$ and we conclude that $(\mathbf{u}_1, \psi_1, c_1) = (\mathbf{u}_2, \psi_2, c_2)$. Therefore the solution of the problem (2.11) is unique. \square

Chapter 3

Numerical Implementation, Stability and Error Analysis

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3.1 Introduction

This chapter is dedicated to the numerical simulations, stability and convergence of the numerical scheme developed to solve the problems (1.101) and (1.103). We shall explain in detail the numerical scheme for the problem (1.103), whereas for the problem (1.101), the numerical scheme can be modified accordingly by including the operator $A_4(\eta, \eta', \eta'', \psi, \nabla\psi, \nabla(\nabla\psi))$ arising due to the anisotropy factor η in the equation (1.101c) of the problem (1.101). To study the convergence and stability, we have added functions $\mathbf{F}_u(\mathbf{x}, t)$, $F_\psi(\mathbf{x}, t)$ and $F_c(\mathbf{x}, t)$ on the right-hand-side of the both problems. We choose the values of the constants (see Table 3.1) for the phase-field and concentration equations in our models as given in [24] and the constants associated with the flow equations are chosen by keeping in view the properties of substances A (Copper (*Cu*) in the present case) and B (Nickel (*Ni*) in the present case). We have

Property Name	Symbol	Unit	Nickel (A)	Copper (B)
Melting temperature	T_m	K	1728	1358
Latent heat	L	J/m^3	2350×10^6	1758×10^6
Diffusion coeff. liquid	D_L	m^2/s	10^{-9}	10^{-9}
Diffusion coeff. solid	D_S	m^2/s	10^{-13}	10^{-13}
Linear kinetic coeff.	β	$m/K/s$	3.3×10^{-3}	3.9×10^{-3}
Interface thickness	δ	m	8.4852×10^{-8}	6.0120×10^{-8}
Density	ρ	Kg/m^3	7810	8020
viscosity	μ	$Pa \cdot s$	4.110×10^{-6}	0.597×10^{-6}
Surface energy	σ	J/m^2	0.37	0.29
Electrical conductivity	σ_e	S/m	14.3×10^6	59.6×10^6
Molar volume	V_m	m^3	7.46×10^{-6}	7.46×10^{-6}
Magnetic-field	B_0	<i>Tesla</i>	100	100
Mode Number	k	N/A	4	4
Anisotropy Amplitude	γ_0	N/A	0.04	0.04

Table 3.1: Physical values of constants

dealt with several examples with known exact solution to study the convergence and stability of the numerical scheme (developed in the next section) for both problems (1.101) and (1.103). We shall present only two examples for the isotropic problem (1.103) and one example for the anisotropic problem (1.101) to validate our approach. In the section (3.2), we shall explain in detail the numerical scheme and give the

space discretization and the general form of the differential-algebraic systems for both problems (1.101) and (1.103). Then we present briefly the time discretization of these problems (details can be found in appendix A). In the section (3.3), we shall explicate the implementation of the developed scheme in the computer softwares Comsol Multiphysics and MatLab. In the next section 3.4, we shall provide examples for the isotropic case (1.103) and anisotropic case (1.101) to validate our approach.

3.2 Discretization of the Problem

This section elucidates the discretization of the problems (1.101) and (1.103), both in time and space. We shall give details of space discretization only for the problem (1.103) whereas the space discretization of the problem (1.101) can be obtained in a similar manner by including the operator $A_4(\eta, \eta', \eta'', \psi, \nabla\psi, \nabla(\nabla\psi))$ arising due to the anisotropy factor η .

For this, we define the variational formulation of the problem (1.103) as

$$\left. \begin{aligned} & \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + a_u(\mathbf{u}, \mathbf{v}) + b_u(\mathbf{u}; \mathbf{u}, \mathbf{v}) + c_p(\mathbf{v}, p) - (\mathbf{A}_1(\psi, c), \mathbf{v}) \\ & \quad - (b(\psi)((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}), \mathbf{v}) = (\mathbf{F}_u, \mathbf{v}), \\ & -c_p(\mathbf{u}, q) = 0, \\ & \left(\frac{\partial \psi}{\partial t}, \phi \right) + a_\psi(\psi, \phi) + b_\psi(\mathbf{u}, \psi, \phi) + (A_2(\psi, c), \phi) = (F_\psi, \phi), \\ & \left(\frac{\partial c}{\partial t}, z \right) + b_c(\mathbf{u}, c, z) + (D(\psi) \nabla c, \nabla z) \\ & \quad + (A_3(\psi, c) \nabla \psi, \nabla z) = (F_c, z), \quad \forall (\mathbf{v}, q, \phi, z) \in \mathcal{W} \times \mathcal{H} \times \mathcal{M} \times \mathcal{M}, \\ & (\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0), \end{aligned} \right\} \quad (3.1)$$

where $c_p(\mathbf{u}, p) = -(\operatorname{div}(\mathbf{u}), p)$ and \mathcal{W} , \mathcal{H} and \mathcal{M} are defined as

$$\mathcal{W} = \left\{ \mathbf{v} \in (H^1(\Omega))^2 \mid \mathbf{v} = 0 \text{ on } \Gamma \right\}, \quad \mathcal{M} = H^1(\Omega) \quad (3.2)$$

$$\mathcal{H} = \left\{ q \in L^2(\Omega) \mid \int_{\Omega} q d\mathbf{x} = 0 \right\}. \quad (3.3)$$

Remark: The condition $\int_{\Omega} q d\mathbf{x} = 0$ on the pressure is imposed in order to assure the uniqueness of the pressure because the pressure is defined within a class of equivalence, regardless of a time-dependent function. We can impose also other conditions on the pressure, in accordance on its regularity, e.g., the pressure is zero on part of

the boundary, etc.

For the discretization of the problem (3.1) with respect to time t , we have used back-ward difference Euler formula and for the space discretization we have used the mixed finite elements for the flow equations and usual finite elements for the concentration and phase-field equations. First, we shall give the space discretization and then the time discretization of the problem (3.1).

Let h be a parameter of discretization such that $0 < h < h_0 < 1$ and let \mathcal{W}_h , \mathcal{H}_h and \mathcal{M}_h are the finite element subspaces of \mathcal{W} , \mathcal{H} and \mathcal{M} respectively associated with the partition \mathcal{T}_h of the domain Ω and the polynomials \mathbb{P}_l , \mathbb{P}_{l-1} and \mathbb{P}_l where l is the degree of the polynomials. We assume that the following conditions hold (see e.g., [47])

(C1) $\exists c_1, \forall \mathbf{X} = (\mathbf{u}, \psi, c) \in (H^{r+1}(\Omega))^4 \cap (\mathcal{W} \times \mathcal{M}^2)$ and $\forall r \in [1, l]$

$$\inf_{\mathbf{X}_h \in \mathcal{W}_h \times \mathcal{M}_h^2} \|\mathbf{X} - \mathbf{X}_h\| \leq c_1 h^r \|\mathbf{X}\|_{H^{r+1}(\Omega)}.$$

(C2) $\exists c_2, \forall q \in H^r(\Omega) \cap \mathcal{H}$ and $\forall r \in [1, l]$

$$\inf_{q_h \in \mathcal{H}_h} \|q - q_h\| \leq c_2 h^r \|q\|_{H^r(\Omega)}.$$

(C3) $\exists c_3$ such that (*InfSup* condition)

$$\inf_{q_h \in \mathcal{H}_h} \sup_{\mathbf{v}_h \in \mathcal{W}_h} \frac{c_p(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\| \|q_h\|} \geq c_3. \quad (3.4)$$

(C4) Let $\mathbf{X}_{0h} = (\mathbf{u}_{0h}, \psi_{0h}, c_{0h})$ be the approximation of $\mathbf{X}_0 = (\mathbf{u}_0, \psi_0, c_0)$ in $\mathcal{W}_h \times \mathcal{M}_h^2$, if $\mathbf{X}_0 \in (H^{r+1}(\Omega))^4$ with $r \in [1, l]$, then

$$h \|\mathbf{X}_0 - \mathbf{X}_{0h}\| + |\mathbf{X}_0 - \mathbf{X}_{0h}| \leq c_4 h^{r+1}.$$

(C5) For all $\mathbf{m}, \mathbf{p}, \mathbf{q}$ and \mathbf{k} integers and $\forall K \in \mathcal{T}_h$ with $0 < \mathbf{p}, \mathbf{q} \leq \infty$, we have

$$\begin{aligned} \|\mathbf{X}_h\|_{W^{\mathbf{m}, \mathbf{q}}(K)} &\leq c_4 h^{n/\mathbf{q} - n/\mathbf{p} + \mathbf{k} - \mathbf{m}} \|\mathbf{X}_h\|_{W^{\mathbf{k}, \mathbf{p}}(K)}, \quad \forall \mathbf{X}_h \in \mathcal{W}_h \times \mathcal{M}_h^2, \\ \|\mathbf{X}_h\|_{W^{\mathbf{m}, \mathbf{q}}(\Omega)} &\leq c_4 h^{n/\mathbf{q} - n/\mathbf{p} + \mathbf{k} - \mathbf{m}} \|\mathbf{X}_h\|_{W^{\mathbf{k}, \mathbf{p}}(\Omega)}, \quad \forall \mathbf{X}_h \in \mathcal{W}_h \times \mathcal{M}_h^2. \end{aligned}$$

We define the space discretization of the problem (3.1) as follows.

Find $(\mathbf{u}_h, p_h, \psi_h, c_h) \in \mathcal{W}_h \times \mathcal{H}_h \times \mathcal{M}_h \times \mathcal{M}_h$ such that $\forall (\mathbf{v}_h, q_h, \varphi_h, z_h) \in \mathcal{W}_h \times \mathcal{H}_h \times$

$$\mathcal{M}_h \times \mathcal{M}_h,$$

$$\begin{aligned} \rho_0 \left(\frac{\partial \mathbf{u}_h}{\partial t}, \mathbf{v}_h \right) + a_u(\mathbf{u}_h, \mathbf{v}_h) + b_u(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + c_p(\mathbf{v}_h, p_h) - (\mathbf{A}_1(\psi_h, c_h), \mathbf{v}_h) \\ - (b(\psi_h)((\mathbf{u}_h \times \mathbf{B}) \times \mathbf{B}), \mathbf{v}_h) = (\mathbf{F}_u, \mathbf{v}_h), \end{aligned} \quad (3.5)$$

$$-c_p(\mathbf{u}_h, q_h) = 0, \quad (3.6)$$

$$\left(\frac{\partial \psi_h}{\partial t}, \varphi_h \right) + a_\psi(\psi_h, \varphi_h) + b_\psi(\mathbf{u}_h, \psi_h, \varphi_h) + (A_2(\psi_h, c_h), \varphi_h) = (F_\psi, \varphi_h), \quad (3.7)$$

$$\begin{aligned} \left(\frac{\partial c_h}{\partial t}, z_h \right) + b_c(\mathbf{u}_h, c_h, z_h) + (D(\psi_h) \nabla c_h, \nabla z_h) \\ + (A_3(\psi_h, c_h) \nabla \psi_h, \nabla z_h) = (F_c, z_h), \end{aligned} \quad (3.8)$$

with the initial condition

$$(\mathbf{u}_h, \psi_h, c_h)(t=0) = (\mathbf{u}_{0h}, \psi_{0h}, c_{0h}), \quad \text{in } \Omega \quad (3.9)$$

Let φ_{ih} for $1 \leq i \leq M$, q_{ih} for $2M+1 \leq i \leq 2M+N$ and z_{ih} for $2M+N+1 \leq i \leq 2M+N+\tilde{M}$ constitutes the basis of \mathcal{W}_h , \mathcal{H}_h and \mathcal{M}_h respectively and

$$\left. \begin{aligned} \mathbf{u}_h &= \sum_{i=1}^M \mathbf{u}_{ih} \varphi_{ih} = \sum_{i=1}^M u_{ih} \underline{\varphi}_{ih}^u + \sum_{i=1}^M v_{ih} \underline{\varphi}_{ih}^v, \\ p_h &= \sum_{i=2M+1}^{2M+N} p_{ih} q_{ih}, \\ \psi_h &= \sum_{i=2M+N+1}^{2M+N+\tilde{M}} \psi_{ih} z_{ih}, \\ c_h &= \sum_{i=2M+N+\tilde{M}+1}^{2M+N+2\tilde{M}} c_{ih} z_{ih}. \end{aligned} \right\} \quad (3.10)$$

where

$$\mathbf{u}_{ih} = \begin{pmatrix} u_{ih} \\ v_{ih} \end{pmatrix}, \quad \underline{\varphi}_{ih}^u = \begin{pmatrix} \varphi_{ih} \\ 0 \end{pmatrix}, \quad \underline{\varphi}_{ih}^v = \begin{pmatrix} 0 \\ \varphi_{ih} \end{pmatrix}$$

Substituting equation (3.10) in the equations (3.5)-(3.8) and simplifying, we arrive at

$$\begin{aligned}
& \sum_{i=1}^M \rho_0 \left(\underline{\varphi}_{ih}^u, \underline{\varphi}_{jh}^u \right) \frac{du_{ih}}{dt} + \sum_{i=1}^M \left\{ a_u \left(\underline{\varphi}_{ih}^u, \underline{\varphi}_{jh}^u \right) + b_u \left(\mathbf{u}_h; \underline{\varphi}_{ih}^u, \underline{\varphi}_{jh}^u \right) \right. \\
& \quad - \left(b(\psi_h) ((\underline{\varphi}_{ih}^u \times \mathbf{B}) \times \mathbf{B}), \underline{\varphi}_{jh}^u \right) \left. \right\} u_{ih} + \sum_{i=2M+1}^{2M+N} \left(q_{ih}, \operatorname{div}(\underline{\varphi}_{jh}^u) \right) p_{ih} \\
& \quad - \left(\mathbf{A}_1(\psi_h, c_h), \underline{\varphi}_{jh}^u \right) \\
& \quad + \sum_{i=1}^M \rho_0 \left(\underline{\varphi}_{ih}^v, \underline{\varphi}_{jh}^v \right) \frac{dv_{ih}}{dt} + \sum_{i=1}^M \left\{ a_u \left(\underline{\varphi}_{ih}^v, \underline{\varphi}_{jh}^v \right) + b_u \left(\mathbf{u}_h, \underline{\varphi}_{ih}^v, \underline{\varphi}_{jh}^v \right) \right. \\
& \quad - \left(b(\psi_h) ((\underline{\varphi}_{ih}^v \times \mathbf{B}) \times \mathbf{B}), \underline{\varphi}_{jh}^v \right) \left. \right\} v_{ih} + \sum_{i=2M+1}^{2M+N} \left(q_{ih}, \operatorname{div}(\underline{\varphi}_{jh}^v) \right) p_{ih} \\
& \quad - \left(\mathbf{A}_1(\psi_h, c_h), \underline{\varphi}_{jh}^v \right) = \left(\mathbf{F}_u, \underline{\varphi}_{jh}^u + \underline{\varphi}_{jh}^v \right), \quad 1 \leq j \leq M \\
& \quad - \sum_{i=2M+1}^{2M+N} \left\{ \left(\operatorname{div}(\underline{\varphi}_{ih}^u), q_{jh} \right) u_{ih} + \left(\operatorname{div}(\underline{\varphi}_{ih}^v), q_{jh} \right) v_{ih} \right\} = 0, \quad 2M+1 \leq j \leq 2M+N \\
& \quad \sum_{i=2M+N+1}^{2M+N+\tilde{M}} (z_{ih}, z_{jh}) \frac{d\psi_{ih}}{dt} + \sum_{i=2M+N+1}^{2M+N+\tilde{M}} \left\{ a_\psi(z_{ih}, z_{jh}) + b_\psi(\mathbf{u}_h, z_{ih}, z_{jh}) \right\} \psi_{ih} \\
& \quad + (A_2(\psi_h, c_h), z_{jh}) = (F_\psi, z_{jh}), \quad 2M+N+1 \leq j \leq 2M+N+\tilde{M} \\
& \quad \sum_{i=2M+N+\tilde{M}+1}^{2M+N+2\tilde{M}} (z_{ih}, z_{jh}) \frac{dc_{ih}}{dt} + \sum_{i=2M+N+\tilde{M}+1}^{2M+N+2\tilde{M}} \left\{ b_c(\mathbf{u}_h, z_{ih}, z_{jh}) \right. \\
& \quad \left. + (D(\psi_h) \nabla z_{ih}, \nabla z_{jh}) \right\} c_{ih} + \sum_{i=2M+N+1}^{2M+N+\tilde{M}} (A_3(\psi_h, c_h) \nabla z_{ih}, \nabla z_{jh}) \psi_{ih} \\
& \quad = (F_c, z_{jh}), \quad 2M+N+\tilde{M}+1 \leq j \leq 2M+N+2\tilde{M}
\end{aligned}$$

The above equations can be written as

$$\mathcal{M} \frac{dY_h}{dt} + \mathcal{A}(Y_h) Y_h + \mathcal{L}(Y_h) = \mathcal{R}, \quad Y_h(t=0) = Y_{0h}, \quad (3.11)$$

with

$$Y_h = (\mathbf{u}_{1h}, \dots, \mathbf{u}_{Mh}, p_{1h}, \dots, p_{Nh}, \psi_{1h}, \dots, \psi_{\tilde{M}h}, c_{1h}, \dots, c_{\tilde{M}h})^{trans}, \quad (3.12)$$

where “*trans*” denotes the usual transpose of a matrix and

$$\mathcal{M} = \begin{pmatrix} M_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{pmatrix} \in \mathbb{R}^{2M+N+2\tilde{M}, 2M+N+2\tilde{M}} \quad (3.13)$$

$$\mathcal{A}(Y_h) = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix} \in \mathbb{R}^{2M+N+2\tilde{M}, 2M+N+2\tilde{M}} \quad (3.14)$$

and

$$\begin{aligned} (M_{11})_{ji} &= \rho_0 \left(\underline{\varphi}_{ih}^u, \underline{\varphi}_{jh}^u \right) + \rho_0 \left(\underline{\varphi}_{ih}^v, \underline{\varphi}_{jh}^v \right), \\ (M_{33})_{ji} &= (z_{ih}, z_{jh}), \quad (M_{44})_{ji} = (z_{ih}, z_{jh}). \end{aligned} \quad (3.15)$$

$$\begin{aligned} (A_{11})_{ji} &= a_u \left(\underline{\varphi}_{ih}^u, \underline{\varphi}_{jh}^u \right) + a_u \left(\underline{\varphi}_{ih}^v, \underline{\varphi}_{jh}^v \right) + b_u \left(\mathbf{u}_h; \underline{\varphi}_{ih}^u, \underline{\varphi}_{jh}^u \right) + b_u \left(\mathbf{u}_h, \underline{\varphi}_{ih}^v, \underline{\varphi}_{jh}^v \right) \\ &\quad - \left(b(\psi_h)((\underline{\varphi}_{ih}^u \times \mathbf{B}) \times \mathbf{B}), \underline{\varphi}_{jh}^u \right) - \left(b(\psi_h)((\underline{\varphi}_{ih}^v \times \mathbf{B}) \times \mathbf{B}), \underline{\varphi}_{jh}^v \right), \end{aligned}$$

$$\begin{aligned} (A_{12})_{ji} &= \left(q_{ih}, \operatorname{div}(\underline{\varphi}_{jh}^u) \right) + \left(q_{ih}, \operatorname{div}(\underline{\varphi}_{jh}^v) \right), \\ (A_{21})_{ji} &= \left(\operatorname{div}(\underline{\varphi}_{ih}^u), q_{jh} \right) + \left(\operatorname{div}(\underline{\varphi}_{ih}^v), q_{jh} \right). \end{aligned}$$

$$(A_{33})_{ji} = a_\psi(z_{ih}, z_{jh}) + b_\psi(\mathbf{u}_h, z_{ih}, z_{jh}),$$

$$(A_{43})_{ji} = (A_3(\psi_h, c_h) \nabla z_{ih}, \nabla z_{jh}),$$

$$(A_{44})_{ji} = (D(\psi_h) \nabla z_{ih}, \nabla z_{jh}) + b_c(\mathbf{u}_h, z_{ih}, z_{jh}),$$

and

$$\mathcal{L}(Y_h) = \begin{pmatrix} L_1 & 0 & L_3 & 0 \end{pmatrix}^{trans}, \quad (3.16)$$

with

$$\begin{aligned} (L_1)_j &= \left(\mathbf{A}_1(\psi_h, c_h), \underline{\varphi}_{jh}^u \right) + \left(\mathbf{A}_1(\psi_h, c_h), \underline{\varphi}_{jh}^v \right), \\ (L_3)_j &= (A_2(\psi_h, c_h), z_{jh}), \end{aligned}$$

and

$$\mathcal{R} = \begin{pmatrix} R_1 & 0 & R_3 & R_4 \end{pmatrix}^{trans}, \quad (3.17)$$

with

$$\begin{aligned} (R_1)_j &= \left(\mathbf{F}_u, \underline{\varphi}_{jh}^u \right) + \left(\mathbf{F}_u, \underline{\varphi}_{jh}^v \right), \\ (R_3)_j &= (F_\psi, z_{jh}), \quad (R_4)_j = (F_c, z_{jh}). \end{aligned}$$

Similarly we can derive, after the space discretization of the problem (1.101), the following differential-algebraic system

$$\mathcal{M} \frac{dY_h}{dt} + \tilde{\mathcal{A}}(Y_h) Y_h + \tilde{\mathcal{L}}(Y_h) = \tilde{\mathcal{R}}, \quad Y_h(t=0) = Y_{0h}, \quad (3.18)$$

where the matrices $\tilde{\mathcal{A}}$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ take into account on the non-linear anisotropy differential operator.

The equations (3.11) and (3.18) can be written in general form as

$$F(t, Y_h, \frac{\partial Y_h}{\partial t}) = 0, \quad Y_h(t=0) = Y_{0h}. \quad (3.19)$$

For the resolution of the above equation, we have used the solver DASSL. For the time discretization, we have used back-ward difference Euler's formula and the resulting non-linear systems are solved using Newton method. Further to solve the system of algebraic equations, we employ the usual Gaussian elimination method. For more details about the solver DASSL, see appendix A.

Before studying the convergence and stability of the numerical scheme (3.19), we shall give postulated error estimates used to compare with the numerical error estimates.

3.2.1 Error Estimates

Let \mathbf{Y} be a Banach space, we define the following spaces for $0 < p \leq +\infty$

$$\begin{aligned} \ell^p(0, T_f, \mathbf{Y}) &= \left\{ \mathbf{u} : (t_1, \dots, t_k) \rightarrow \mathbf{Y} \text{ such that } \|\mathbf{u}\|_{\ell^p(0, T_f, \mathbf{Y})} = \left(\tau \sum_{i=1}^k \|\mathbf{u}_i\|_{\mathbf{Y}}^p \right)^{1/p} < \infty \right\} \\ \ell^\infty(0, T_f, \mathbf{Y}) &= \left\{ \mathbf{u} : (t_1, \dots, t_k) \rightarrow \mathbf{Y} \text{ such that } \|\mathbf{u}\|_{\ell^\infty(0, T_f, \mathbf{Y})} = \max_{1 \leq i \leq k} \|\mathbf{u}_i\|_{\mathbf{Y}} < \infty \right\} \end{aligned}$$

where $\mathbf{u}_i = \mathbf{u}(t_i)$ and $t_k = T_f$.

We postulate that the error estimates obtained by solving the problem (3.1) using numerical scheme (3.19) are as below

$$\|\Psi_h - \Psi\|_{\ell^2(0, T_f, L^2(\Omega))} \leq C(\tau^\alpha + h^{\beta_1}) \quad (3.20)$$

$$\|p_h - p\|_{\ell^2(0, T_f, L^2(\Omega))} \leq C(\tau^\alpha + h^{\beta_2}) \quad (3.21)$$

where $\Psi_h = (\mathbf{u}_h, \psi_h, c_h)$ is the numerical approximate solution and $\Psi = (\mathbf{u}, \psi, c)$ is the known exact solution of the considered problem. These formulas are of order α in time and of order β_1 and β_2 in space respectively, where β_i , $i = 1, 2$, are greater than 1 and less than minimum of the degree of the finite elements (polynomials) and

the Sobolev space regularity of the solutions. Same type of error estimates can be found in [3] and [17]. To achieve a reasonable convergence rate with respect to spatial coordinates (*i.e.*, β_1, β_2) we have to take $\tau \leq h^{\beta_i}$, $i = 1, 2$ and for the convergence rate with respect to time (*i.e.*, α), we have to choose $h^{\beta_i} \leq \tau$, $i = 1, 2$ to attain the optimal precision. To study the convergence of the numerical scheme (3.19), we shall perform two type of computations. First, we shall compute the numerical convergence error with respect to spatial step h using the finite elements as polynomials \mathbb{P}_2 and \mathbb{P}_3 of degree 2 and 3 respectively. According to theoretical postulated errors (3.20) and (3.21), we should obtain the error estimates $\beta_i = 3, 4$, $i = 1, 2$ for the \mathbb{P}_2 and \mathbb{P}_3 respectively. Next, we shall calculate numerically the convergence rate with respect to time step τ (using back-ward difference Euler formula). In this case the order of convergence should be equal to 1 to verify that the convergence rates coincide with the estimates speculated in (3.20) and (3.21).

3.3 Implementation details

This section elaborates the implementation of the numerical scheme (3.19) used to solve problems (1.101) and (1.103) in Comsol Multiphysics 3.4 together with Matlab 2007a. Comsol Multiphysics is a simulations software that can be used to solve steady and time dependent as well as linear and non-linear PDEs using finite element method in 1 to 3 space dimensions and Lagrange elements of degree 1 to 3. The choice of this package is motivated as it provides an interface with the MatLab to utilize its graphical user interface with a lot of flexibility in mesh generation.

As described earlier in the section 3.2, we have used back-ward Euler's difference formula for the time discretization and the resulting non-linear fixed point systems are then solved by Newton method. And for the space discretization we have used mixed finite elements which satisfy the *InfSup* condition (Babuska-Brezis condition) (3.4) for the Navier-Stokes type system and the usual finite elements for phase-field and concentration equations. In order to solve the obtained non-linear differential-algebraic system, we have used the solver DASSL (for more details about DASSL, see appendix A).

Further in Comsol Multiphysics 3.4, we have used general Navier-Stokes equations transient analysis mode together with the no-slip boundary conditions to develop the magnetohydrodynamic systems by introducing the magnetic-field. Diffusion-convection transient modes together with convective flux boundary conditions are used to introduce the phase-field and concentration equations respectively of the problem

(3.1) in two dimensions. Two kinds of finite elements (\mathbb{P}_2 and \mathbb{P}_3) are used to study the convergence of the model alongwith the sequence of unstructured meshes which are elaborated below.

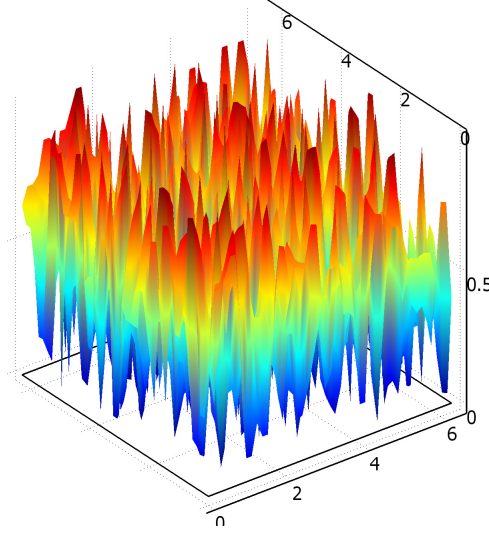
To check the stability of the numerical scheme (3.19), we have multiplied the right-hand-side of each equation of problems (1.101) and (1.103) by a random function $randfn = (1 - \epsilon randf)$ whose value varies between 0 and 1 and ϵ is the parameter used to fix the percentage of random error. This function creates perturbation in the numerical scheme, we have verified the stability of the numerical scheme by increasing the percentage of random error up to 40%. The 3D plot of random function $randfn$ is given in the Fig. 3.1. And to study the stability, \mathbb{P}_2 finite elements are used for the velocity \mathbf{u} , phase-field variable ψ and concentration c and \mathbb{P}_1 finite elements are used for the pressure p .

The implementation of the numerical scheme in Comsol to study the convergence and stability of the models (1.101) and (1.103) is not evident, especially to introduce the anisotropy function, random function and by considering the real physical parameters, we have used some of the MatLab functions. To study the convergence of the numerical scheme for both models with respect to spatial and time coordinates, we have written computer programs in MatLab (connected with the Comsol) to implement a Loop for the successive changes in the spatial and time steps to obtain the convergence rates and the corresponding error curves.

In the next section, we shall present convergence and stability of the numerical scheme (3.19) for the problems (1.101) and (1.103) by considering various examples with known exact solutions.

3.4 Numerical Examples: Error Estimates and Stability

We present in this section, the convergence and stability of the numerical scheme (3.19) by considering examples with known exact solutions. In order to validate our approach for the isotropic and anisotropic cases (1.101) and (1.103) respectively, first we shall present two examples for the isotropic case and then we shall give one example for the anisotropic case.

Figure 3.1: Random function *randfn*.

3.4.1 Isotropic Case: Example 1

For simplicity, we assume that the final time is $T_f = 1$, unless otherwise specified. The domain is a square region $\Omega = [0, 2\pi] \times [0, 2\pi]$ in \mathbb{R}^2 . We have considered the exact solution of the problem (1.103) as

$$\begin{aligned} u_{ex}(x, y, t) &= \frac{2}{(2\pi)^2} e^{1-t} \sin(x)^2 y \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right), \\ v_{ex}(x, y, t) &= -\frac{2}{(2\pi)^2} e^{1-t} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2, \\ p_{ex}(x, y, t) &= e^{1-t} \cos(y), \quad \mathbf{B} = \frac{1}{\sqrt{2}}(1, 1), \\ \psi_{ex}(x, y, t) &= \frac{e^{1-t}}{2} (\cos(x) \cos(y) + 1), \\ c_{ex}(x, y, t) &= \frac{8}{(2\pi)^2} e^{1-t} x^2 \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1). \end{aligned}$$

where $\mathbf{u}_{ex} = (u_{ex}, v_{ex})$ and the corresponding data \mathbf{F}_u , F_ψ and F_c is calculated analytically by substituting the exact solution in the problem (3.1). The expressions for \mathbf{F}_u , F_ψ and F_c are given in appendix C. Then we have computed the numerical solution (\mathbf{u}, p, ψ, c) of the problem (3.1) and compared it to the exact solution $(\mathbf{u}_{ex}, p_{ex}, \psi_{ex}, c_{ex})$ given above.

3.4.1.1 Test Meshes

To investigate the convergence of the numerical scheme (3.19), we build a sequence of five meshes with a decreasing step h with respect to spatial coordinates (see Fig. 3.2). The mesh statistics are given in the following Table 3.2. where No. of B. elements is

Domain No.	step size h	No. of Elements	No. of B. elements
1	0.8	270	8
2	0.7	338	9
3	0.6	534	11
4	0.4	1082	16
5	0.2	4380	32

Table 3.2: Mesh Statistics

abbreviated for the number of boundary elements.

3.4.1.2 Error Analysis

We shall present now the experiments made to investigate the performance of the method. We have performed two types of computations to check the convergence of the numerical scheme (3.19).

The first is to check the spatial convergence rate, in which, a small time step τ is fixed as compared to the spatial step size h and we have varied the spatial step size h as described in the Table 3.2 of mesh statistics. To calculate the rates β_1 and β_2 , we have used $\mathbb{P}_2 - \mathbb{P}_1$ and $\mathbb{P}_3 - \mathbb{P}_2$ mixed finite elements for the velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ respectively and for the phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$, we have used \mathbb{P}_2 and \mathbb{P}_3 , where \mathbb{P}_l , $l = 2, 3$ is the polynomial of degree l . The estimates for β_1 and β_2 are given in the Table 3.3 and Table 3.4 respectively. And the corresponding error curves for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are given in the Fig. 3.3. In Fig. 3.3, we give the plots of

—	$\mathbb{P}_2 - \mathbb{P}_1$	$\mathbb{P}_3 - \mathbb{P}_2$
β_1 for \mathbf{u}	2.6201	3.8730
β_2 for p	1.9207	3.0646

Table 3.3: Order of convergence β_1 for \mathbf{u} and β_2 for p .

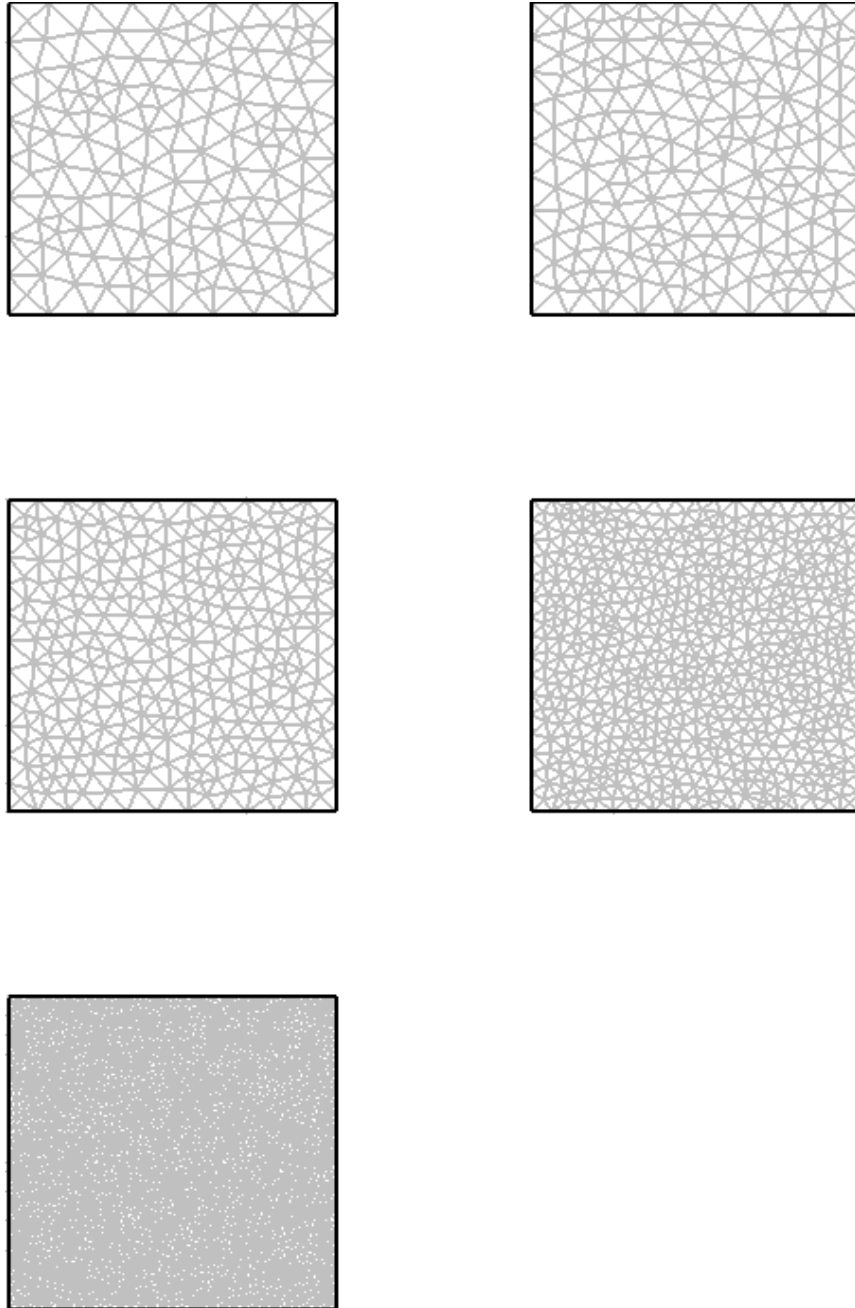


Figure 3.2: Meshes used for Convergence study.

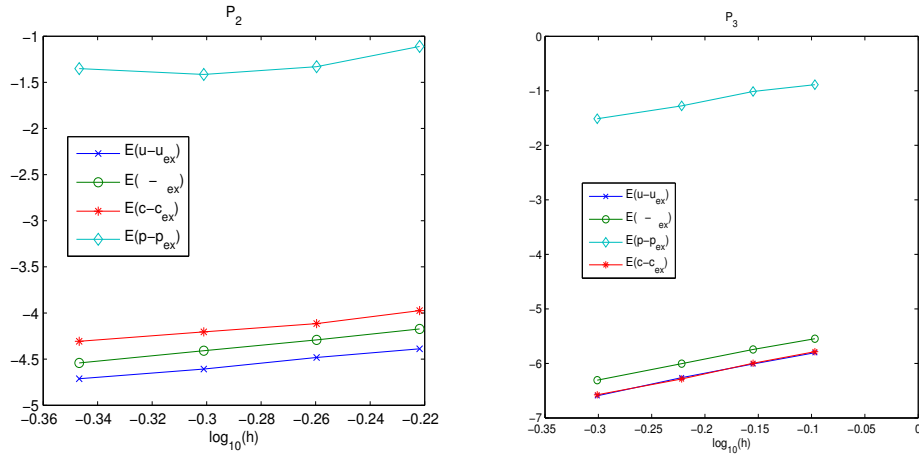


Figure 3.3: Error curves of norm $L_2(\mathcal{Q})$ for the velocity \mathbf{u} , pressure p , phase-field ψ and concentration c with respect to h .

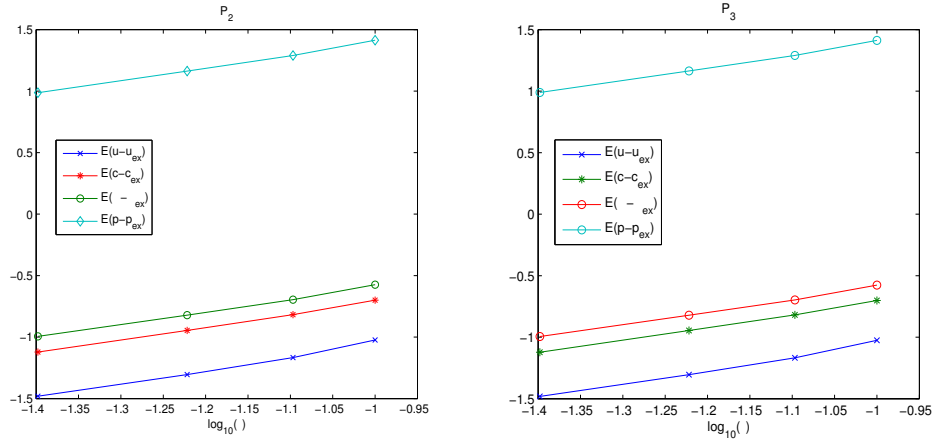


Figure 3.4: Error curves of norm $L_2(\mathcal{Q})$ for the velocity \mathbf{u} , pressure p , phase-field ψ and concentration c with respect to t .

—	\mathbb{P}_2	\mathbb{P}_3
β_1 for ψ	2.7001	3.7501
β_1 for c	2.9278	3.8739

Table 3.4: Order of convergence β_1 for ψ and c .

the \log_{10} of the norm $L_2(\mathcal{Q})$ of velocity \mathbf{u} , pressure p , phase field ψ and concentration

c versus \log_{10} of the step h to obtain the approximations of the parameters β_1 and β_2 given in the formulas (3.20) and (3.21). We observe that the slopes of the error curves for the velocity \mathbf{u} , phase-field ψ and concentration c are approximately equal to 3 and 4 in case of \mathbb{P}_2 and \mathbb{P}_3 finite elements respectively and the slopes of the error curves for the pressure p are approximately equal to 2 and 3 in case of \mathbb{P}_1 and \mathbb{P}_2 finite elements respectively. This shows that our numerical error estimates agree with the theoretical postulated error estimates given by (3.20) and (3.21).

The second computations have been made to test the temporal convergence rate, in which, a small spatial step h is fixed and the convergence rate α is computed with respect to t for the same pattern of the finite elements as we have used in the first case. The estimates for α with respect to time are given in the Table 3.5. And the corresponding error curves for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are given in the Fig. 3.4. The Fig. 3.4 shows the plots of the

—	$\mathbb{P}_2 - \mathbb{P}_1$	$\mathbb{P}_3 - \mathbb{P}_2$
for \mathbf{u}	1.1494	1.1446
for ψ	1.0558	1.0496
for c	1.0602	1.0565
for p	1.0733	1.0634

Table 3.5: Order of convergence α .

\log_{10} of the norm $L_2(\mathcal{Q})$ of velocity \mathbf{u} , pressure p , phase field ψ and concentration c versus \log_{10} of the step τ to obtain an approximation of the convergence rate α with respect to time. We note that the slopes of the error curves for the velocity \mathbf{u} , phase-field ψ and concentration c are nearly equal to 1 in case of \mathbb{P}_2 and \mathbb{P}_3 finite elements respectively and the slopes of the norm $L_2(\mathcal{Q})$ of the pressure p are also approximately equal to 1 in case of \mathbb{P}_1 and \mathbb{P}_2 elements respectively. From this we conclude that the numerical error estimates are in good agreement with the theoretical postulated errors defined by (3.20) and (3.21).

3.4.1.3 Stability Analysis

To study the stability of our model, as mentioned earlier in the section (3.3), we have multiplied the right hand side terms $F_{\mathbf{u}}, F_{\psi}$ and F_c by a term $(1 - \epsilon \text{ randfn})$ to create perturbations in the numerical scheme (3.19), where randfn is a random function which takes values in the interval $[0, 1]$ and ϵ is a parameter which is used to fix the percentage of the random error. The three dimensional plot of the random function

randfn is given in the Fig. 3.1. To verify that our model is stable against perturbations, we have performed three types of computations. First, for different values of the ϵ , we have computed the norm $L_2(\mathcal{Q})$ between the exact solution, denoted by $\Phi_{ex} = (\mathbf{u}_{ex}, p_{ex}, \psi_{ex}, c_{ex})$, and the perturbed solution, denoted by $\Phi_\epsilon = (\mathbf{u}_\epsilon, p_\epsilon, \psi_\epsilon, c_\epsilon)$, that is

$$E(\Phi_\epsilon - \Phi_{ex}) = \|\Phi_\epsilon - \Phi_{ex}\|_{L_2(\mathcal{Q})} \quad (3.22)$$

In Fig. 3.5, the curves of the Norm $L_2(\mathcal{Q})$, defined by (3.22), for the velocity \mathbf{u} ,

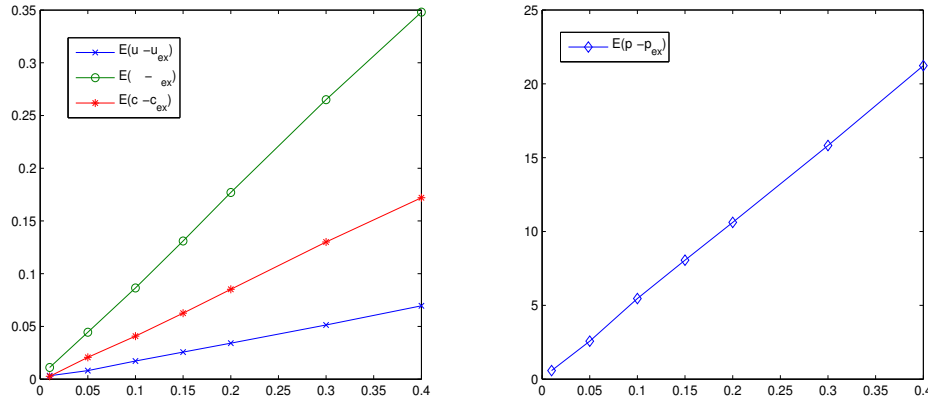


Figure 3.5: Error curves of $E(\Phi_\epsilon - \Phi_{ex})$ versus ϵ .

pressure p , phase-field variable ψ and concentration c versus ϵ are shown for $\epsilon = 0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4$. We found that the error curves are straight lines with linear dependence of error with respect to ϵ , i.e.,

$$E(\Phi_\epsilon - \Phi_{ex}) \approx C \epsilon \quad (3.23)$$

where C represents the slopes of the error curves given in the Table 3.6. Second,

Slope	$E(\Phi_\epsilon - \Phi_{ex})$	$E(\Phi_\epsilon - \Phi_{app})$
$m_{\mathbf{u}}$	0.1701	0.1754
m_{ψ}	0.8638	0.8818
m_c	0.4341	0.4375
m_p	52.9718	52.8359

Table 3.6: Slopes of Norm $E(\Phi_\epsilon - \Phi_{ex})$ and $E(\Phi_\epsilon - \Phi_{app})$.

we have computed the norm $L_2(\mathcal{Q})$ between the approximate solution, denoted by $\Phi_{app} = (\mathbf{u}_{app}, p_{app}, \psi_{app}, c_{app})$, and the perturbed solution for different values of the ϵ , that is

$$E(\Phi_\epsilon - \Phi_{app}) = \|\Phi_\epsilon - \Phi_{app}\|_{L_2(\mathcal{Q})} \quad (3.24)$$

Note that approximate solution Φ_{app} is a solution of the model problem (3.1) with out random error i.e., in this case $\epsilon = 0$. The Fig. 3.6 shows the curves of the norm $E(\Phi_\epsilon - \Phi_{app})$, defined by (3.24), of the velocity \mathbf{u} , pressure p , phase-field variable ψ and concentration c versus ϵ for values of the ϵ taken same as in the first case. Again we found that the error curves are straight lines with a linear dependence of error with respect to ϵ as we have found in the first case. The slopes of the error curves are given in Table 3.6. We observe also that the slopes are approximately same for the both cases.

Third, we have solved the model repetitively by increasing the random error and

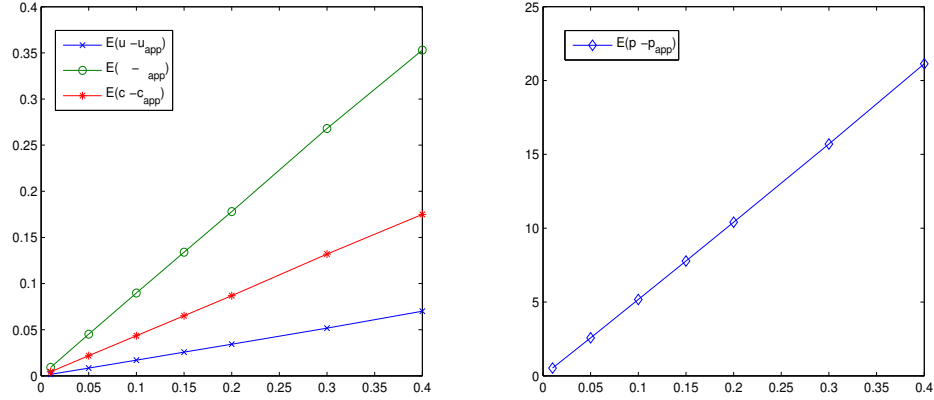
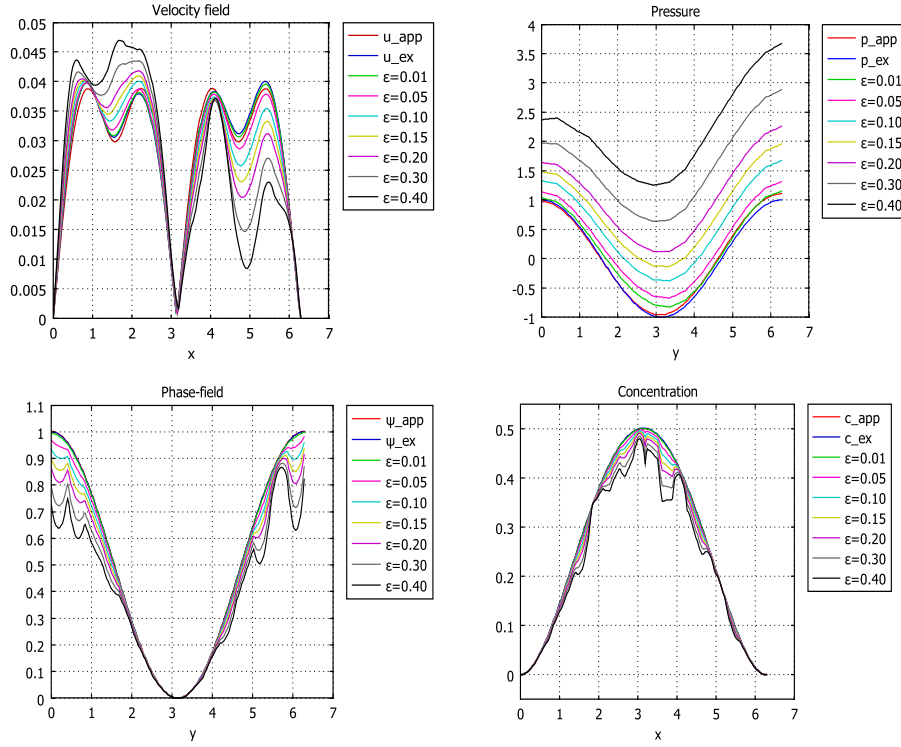


Figure 3.6: Error curves of $E(\Phi_\epsilon - \Phi_{app})$ versus ϵ .

present the solution curves of the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ on a part of the domain for different values of the ϵ to verify that our model is stable against the perturbations generated by a random function *randfn*. We can see in Fig. 3.7 that the solution curves of the model remains stable and it does not become unstable by increasing the random error.

In Fig. 3.7, we have given the solution of the velocity $\mathbf{u}(\mathbf{x}, t)$ at time $t = 1$, $y = \pi/2$ and x varies from 0 to 2π , pressure $p(\mathbf{x}, t)$ at time $t = 1$, $x = 2\pi$ and y varies from 0 to 2π , phase-field $\psi(\mathbf{x}, t)$ at time $t = 1$, $x = 2\pi$ and y varies from 0 to 2π and concentration $c(\mathbf{x}, t)$ at time $t = 1$, $y = \pi/2$ and x varies from 0 to 2π for the different

Figure 3.7: Solution curves for the different values of ϵ .

values of the random error $\epsilon = 0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4$. Also note that the solution corresponding the $\epsilon = 0.00$ is the approximate solution Φ_{app} of the model with out random error.

3.4.2 Isotropic Case: Example 2

We choose another example to verify that the convergence and stability of problem (3.1) is not specific to Example 1. In this example, same type of computations have been made as in the Example 1. Here, the domain is a square region $\Omega = [0, 1] \times [0, 1]$ in \mathbb{R}^2 and the final time is fixed $T_f = 1$. The exact solution is taken as

$$\begin{aligned}
 u_{ex}(x, y, t) &= 4\pi e^{t-1} x^2 (1-x)^2 \sin(2\pi y) \cos(2\pi y), \\
 v_{ex}(x, y, t) &= -2e^{t-1} x(2x^2 - 3x + 1) \sin^2(2\pi y), \\
 p_{ex}(x, y, t) &= e^{t-1} \cos(2\pi x), \quad \mathbf{B} = \frac{1}{\sqrt{2}}(1, 1), \\
 \psi_{ex}(x, y, t) &= \frac{1}{4} e^{t-1} (\cos(2\pi x) + \cos(2\pi y) + 2), \\
 c_{ex}(x, y, t) &= 8e^{t-1} (x^2(1-x)^2 + y^2(1-y)^2).
 \end{aligned}$$

The corresponding data \mathbf{F}_u , F_ψ and F_c is calculated analytically by substituting the exact solution $(\mathbf{u}_{ex}, p_{ex}, \psi_{ex}, c_{ex})$ in the system (3.1). The expressions for \mathbf{F}_u , F_ψ and F_c are given in appendix C.

3.4.2.1 Test Meshes

To study the convergence of the model problem (3.1), a sequence of five meshes with a decreasing spatial step h is considered (see Fig. 3.8). The mesh statistics is given in the Table 3.7, where No. of B. Elements is abbreviated for the number of boundary

Domain No.	step size h	No. of Elements	No. of B. elements
1	0.2	106	5
2	0.15	200	7
3	0.1	434	10
4	0.05	1712	20
5	0.01	42904	100

Table 3.7: Mesh Statistics

elements.

3.4.2.2 Error Analysis

We present here the numerical experiments made to obtain the error estimates defined in (3.20) and (3.21). Like previous Example 1, we have made two types of computations also in this example. The first one is to check the spatial convergence rate, in which, a small time step τ is taken as compared to the spatial step size h which is varied according to the Table 3.7 of mesh statistics. We have calculated the rate β_1 using the \mathbb{P}_2 and \mathbb{P}_3 for the phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ and for the velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$, we have used the $\mathbb{P}_2 - \mathbb{P}_1$ and $\mathbb{P}_3 - \mathbb{P}_2$ finite elements to calculate the convergence rates β_1 and β_2 respectively. Then the estimates for the β_1 and β_2 are calculated and given in the Table 3.8 and Table 3.9 respectively. And the corresponding error curves of the Norm $L_2(\mathcal{Q})$ of the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are given in the Fig. 3.9. In Fig. 3.9, we present the plots of the \log_{10} of the Norm $L_2(\mathcal{Q})$ of the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ versus $\log_{10}(h)$. We observe that the slopes of the error curves of velocity $\mathbf{u}(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are approximately equal to 3 and 4 in case of \mathbb{P}_2 and \mathbb{P}_3 finite elements and the

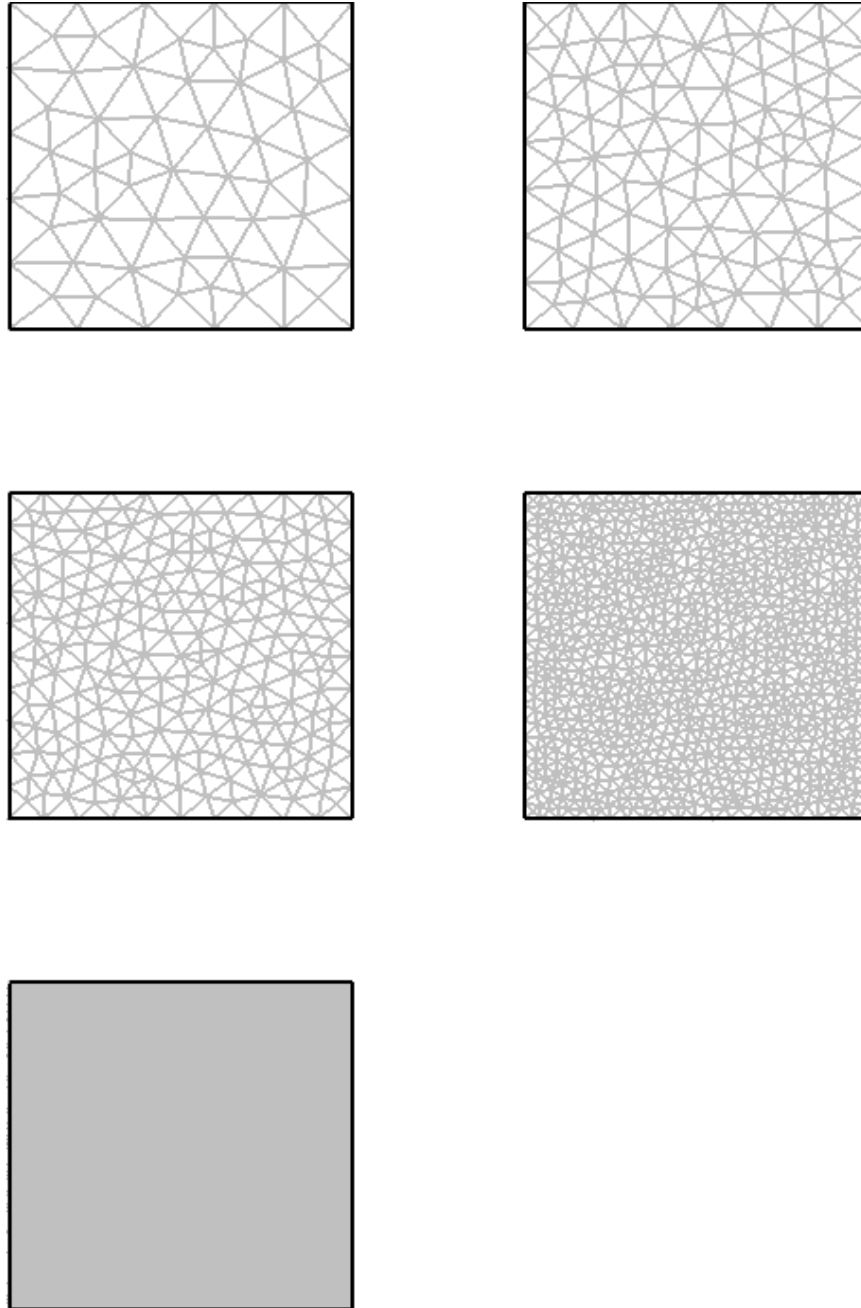


Figure 3.8: Meshes used for Convergence study.

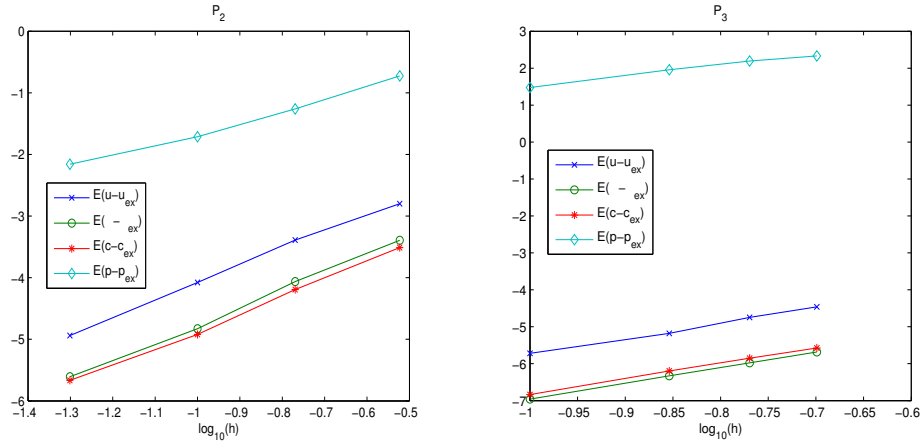


Figure 3.9: Error curves of norm $L_2(\mathcal{Q})$ for the velocity \mathbf{u} , pressure p , phase-field ψ and concentration c with respect to h .

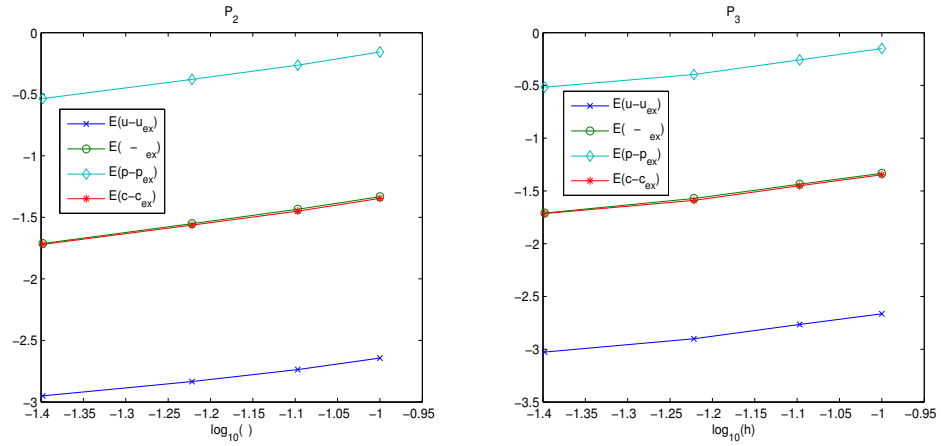


Figure 3.10: Error curves of norm $L_2(\mathcal{Q})$ for the velocity \mathbf{u} , pressure p , phase-field ψ and concentration c with respect to t .

—	$\mathbb{P}_2 - \mathbb{P}_1$	$\mathbb{P}_3 - \mathbb{P}_2$
β_1 for \mathbf{u}	2.7501	4.1783
β_2 for p	1.8426	2.8381

Table 3.8: Order of convergence β_1 for \mathbf{u} and β_2 for p .

slopes of the error curve of the pressure p are approximately equal to 2 and 3 in case

—	\mathbb{P}_2	\mathbb{P}_3
β_1 for ψ	2.8001	4.1868
β_1 for c	2.8449	4.2376

Table 3.9: Order of convergence β_1 for ψ and c .

of \mathbb{P}_1 and \mathbb{P}_2 finite elements respectively. We notice that the numerical error estimates are approximately equal to the postulated error estimates (3.20) and (3.21).

Second is to investigate the convergence rate with respect to time, in which, small

-	$\mathbb{P}_2 - \mathbb{P}_1$	$\mathbb{P}_3 - \mathbb{P}_2$
for \mathbf{u}	0.8501	0.9100
for ψ	0.9377	0.9259
for c	0.9554	0.9456
for p	0.9512	0.9189

Table 3.10: Order of convergence α .

spatial step size h is used as compared to the time steps τ . Same type of finite elements are used as in the first case. The estimates of α are calculated and given in the Table 3.10 and corresponding error curves of the norm $L_2(\mathcal{Q})$ for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are given in the Fig. 3.10. In Fig. 3.10, the plots of \log_{10} of the norm $L_2(\mathcal{Q})$ of the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ versus $\log_{10}(\tau)$ are given. We note that the slopes of the error curves are nearly equal to 1 in both kinds of finite elements which coincide with the theoretical postulated errors in (3.20) and (3.21).

3.4.2.3 Stability Analysis

The stability of the model problem (3.1) is studied also on the same pattern as in Example 1. A random function *randfn*, which takes values between 0 and 1, is used to create perturbations in the model problem and a parameter ϵ is used to fix the percentage of the random error. The plot of the function *randfn* is given in the Fig. 3.1. Here also, we have made three kind of computations to check the stability of our model problem as in Example 1. First, we have computed the Norm $E(\Phi_\epsilon - \Phi_{ex})$ given in equation (3.22). The plots of the curves of Norm $E(\Phi_\epsilon - \Phi_{ex})$ for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ versus ϵ are given

in the Fig. 3.11. Again we observe that the error curves are straight lines with the linear dependence of error with respect to ϵ , i.e.,

$$E(\Phi_\epsilon - \Phi_{ex}) \approx C \epsilon \quad (3.25)$$

where C represents the slopes of the error curves and are given in the Table 3.11.

Secondly, the norm $E(\Phi_\epsilon - \Phi_{app})$ given in equation (3.24) is computed for the velocity

Slope	$E(\Phi_\epsilon - \Phi_{ex})$	$E(\Phi_\epsilon - \Phi_{app})$
m_u	0.0628	0.0628
m_ψ	0.1276	0.1277
m_c	0.1021	0.1028
m_p	1.4738	1.4913

Table 3.11: Slopes of norm $E(\Phi_\epsilon - \Phi_{ex})$ and $E(\Phi_\epsilon - \Phi_{app})$.

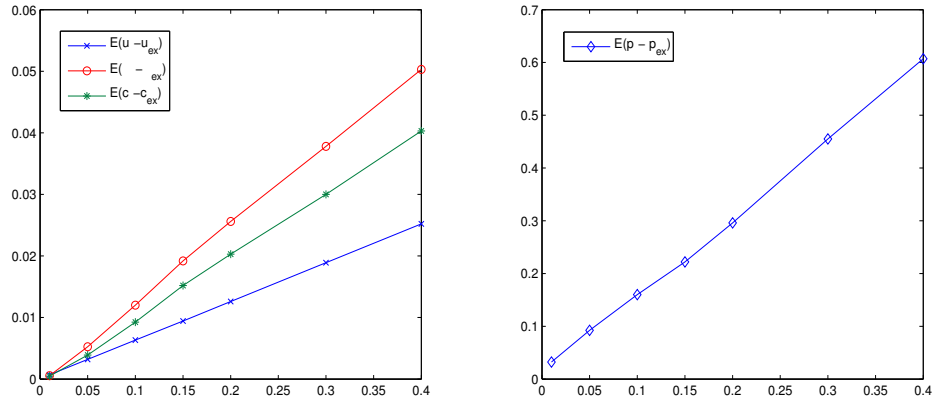


Figure 3.11: Error curves of $E(\Phi_\epsilon - \Phi_{ex})$ versus ϵ .

$\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$. The error curves are given in the Fig. 3.12 for random error same as in the first case, and the slopes of the error curves are given in the Table 3.11. We again found that the error curves are straight lines with the linear dependence of error with respect to ϵ and the slopes in the first and second cases are approximately same.

Third, to check the stability of our model (3.1), we have generated the perturbations in the model with help of a random function *randfn* and solved the model by gradually

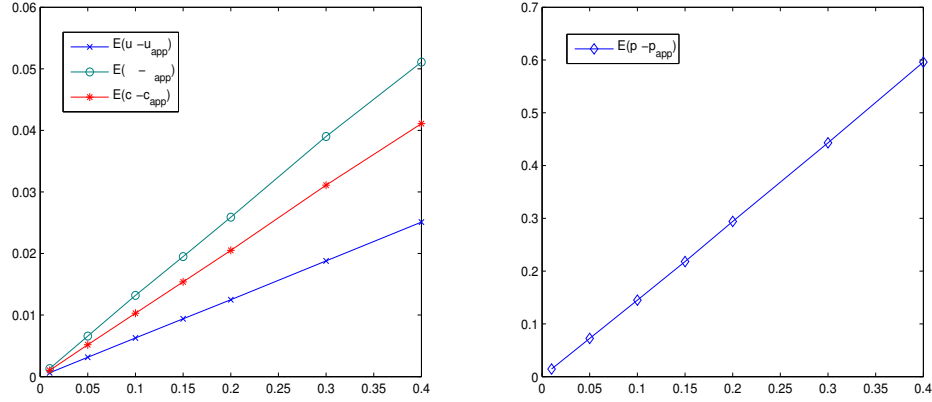
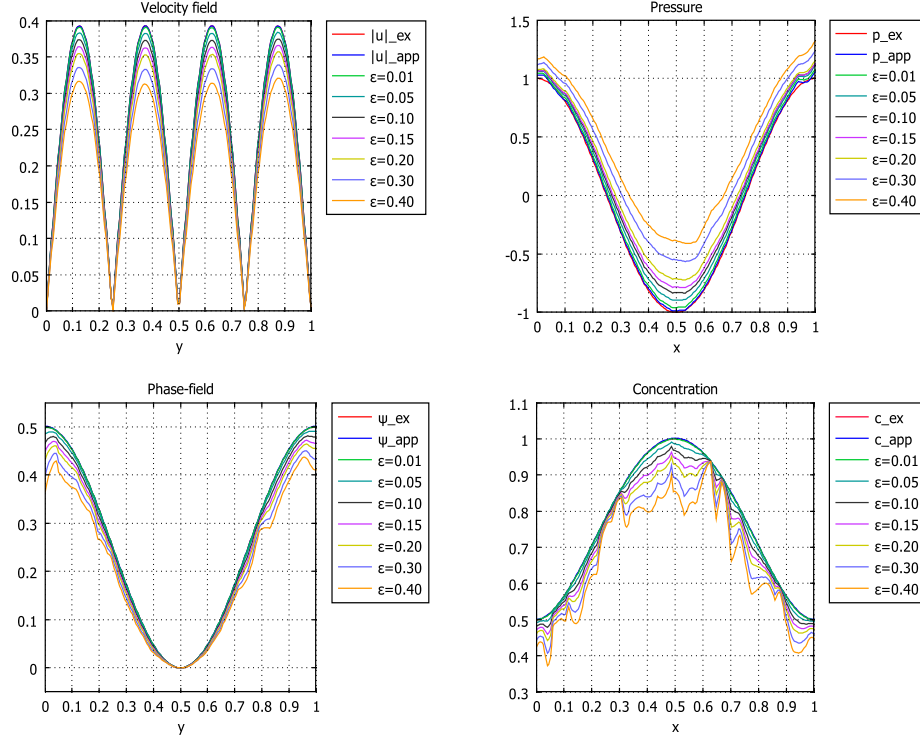


Figure 3.12: Error curves of $E(\Phi_\epsilon - \Phi_{app})$ versus ϵ .

increasing the random error upto 40% and the solution curves of velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ for different values of the ϵ are given in the Fig. 3.13. We observe that as we increase the percentage of the random error, i.e. ϵ , the solution is more perturbed, but it does not become unstable.

In Fig. 3.13, we have given the solution of the velocity $\mathbf{u}(\mathbf{x}, t)$ at time $t = 1$, $x = 1/2$ and y varies from 0 to 1, pressure $p(\mathbf{x}, t)$ at time $t = 1$, $y = 1/2$ and x varies from 0 to 1, phase-field $\psi(\mathbf{x}, t)$ at time $t = 1$, $x = 1/2$ and y varies from 0 to 1 and concentration $c(\mathbf{x}, t)$ at time $t = 1$, $y = 1/2$ and x varies from 0 to 1 for the different values of the random error $\epsilon = 0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4$. Also note that the solution corresponding the $\epsilon = 0.00$ is the approximate solution of the model with out random error.

Figure 3.13: Solution curves for the different values of ϵ .

3.4.3 Anisotropic Case: Example 3

As described earlier, the numerical scheme to solve the problem (1.101) can be given by revising appropriately the scheme (3.19) and introducing the operator $A_4(\eta, \eta', \eta'', \psi, \nabla\psi, \nabla(\nabla\psi))$ arising due to the anisotropy factor η in the equation (1.101c). In this section, we shall give an example of the isothermal-anisotropic model (1.101) to study the convergence and stability of the numerical scheme (3.19). The functions $\mathbf{F}_u(\mathbf{x}, t)$, $F_\psi(\mathbf{x}, t)$ and $F_c(\mathbf{x}, t)$ added on the right hand side in this case will also be modified and are calculated analytically by substituting $(\mathbf{u}_{ex}, p_{ex}, \psi_{ex}, c_{ex})$ in the model (1.101) which are given in the appendix C. In this example, same type of computations have been made as in the Examples 1 and 2. The domain is a square region $\Omega = [0, 1] \times [0, 1]$ in \mathbb{R}^2 and the final time is fixed $T_f = 1$. The exact solution

is taken same as in the Example 2 and is given again by

$$\begin{aligned} u_{ex}(x, y, t) &= 4\pi e^{t-1} x^2 (1-x)^2 \sin(2\pi y) \cos(2\pi y), \\ v_{ex}(x, y, t) &= -2e^{t-1} x (2x^2 - 3x + 1) \sin^2(2\pi y), \\ p_{ex}(x, y, t) &= e^{t-1} \cos(2\pi x), \quad \mathbf{B} = \frac{1}{\sqrt{2}}(1, 1), \\ \psi_{ex}(x, y, t) &= \frac{1}{4} e^{t-1} (\cos(2\pi x) + \cos(2\pi y) + 2), \\ c_{ex}(x, y, t) &= 8e^{t-1} (x^2(1-x)^2 + y^2(1-y)^2). \end{aligned}$$

Same type of meshes, equation modes and boundary conditions have been used to solve the system (1.101) as in the Example 2 (see Example 2 for details).

3.4.3.1 Error Analysis

We present here the numerical experiments made to obtain the error estimates in (3.20) and (3.21). We have calculated the rate β_1 and β_2 which are given in the Table 3.12 and Table 3.13 respectively. And the corresponding error curves of the Norm

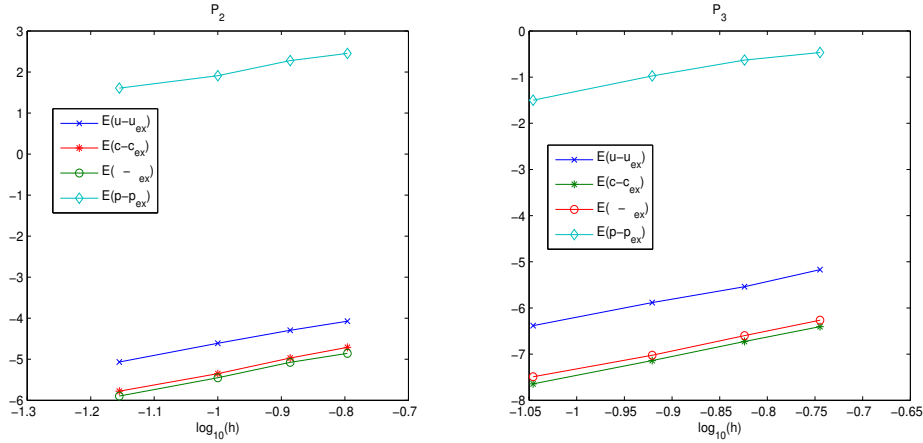


Figure 3.14: Error curves of norm $L_2(\mathcal{Q})$ for the velocity \mathbf{u} , pressure p , phase-field ψ and concentration c with respect to h .

$L_2(\mathcal{Q})$ of the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are given in the Fig. 3.14. The error estimates in this case are also in good accordance with the theoretical postulated error (3.20) and (3.21).

Second, the estimates of α are computed and given in the Table 3.14 and the corresponding error curves of the Norm $L_2(\mathcal{Q})$ for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$,

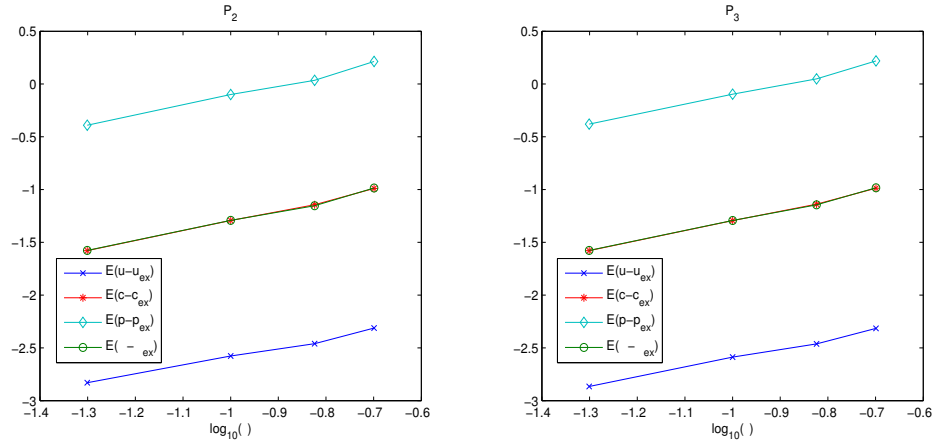


Figure 3.15: Error curves of norm $L_2(\mathcal{Q})$ for the velocity \mathbf{u} , pressure p , phase-field ψ and concentration c with respect to t .

—	$\mathbb{P}_2 - \mathbb{P}_1$	$\mathbb{P}_3 - \mathbb{P}_2$
β_1 for \mathbf{u}	2.7664	4.0303
β_2 for p	2.3462	3.4302

Table 3.12: Order of convergence β_1 for \mathbf{u} and β_2 for p .

—	\mathbb{P}_2	\mathbb{P}_3
β_1 for ψ	2.8972	4.0681
β_1 for c	2.9670	4.1189

Table 3.13: Order of convergence β_1 for ψ and c .

—	$\mathbb{P}_2 - \mathbb{P}_1$	$\mathbb{P}_3 - \mathbb{P}_2$
for \mathbf{u}	0.9011	0.9152
for ψ	0.9821	0.9856
for c	0.9792	0.9815
for p	1.0032	0.9944

Table 3.14: Order of convergence α .

phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are given in the Fig. 3.15. We note that

the slopes of the error curves are nearly equal to 1 in all kinds of finite elements which coincide with the theoretical error estimates in (3.20) and (3.21).

3.4.3.2 Stability Analysis

The stability of the numerical scheme is studied also on the same pattern as in Example 2 (see Example 2 for details). First, we have computed the norm $L_2(\mathcal{Q})$ given in equation (3.22). The plots of the curves of norm $L_2(\mathcal{Q})$ for the velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$ are given in the Fig. 3.16. Again we observe that the error curves are straight lines with the linear dependence of error with respect to ϵ and the slopes of these lines are given in the Table 3.15.

Second, the norm $L_2(\mathcal{Q})$ given in equation (3.24) is computed for the velocity $\mathbf{u}(\mathbf{x}, t)$,

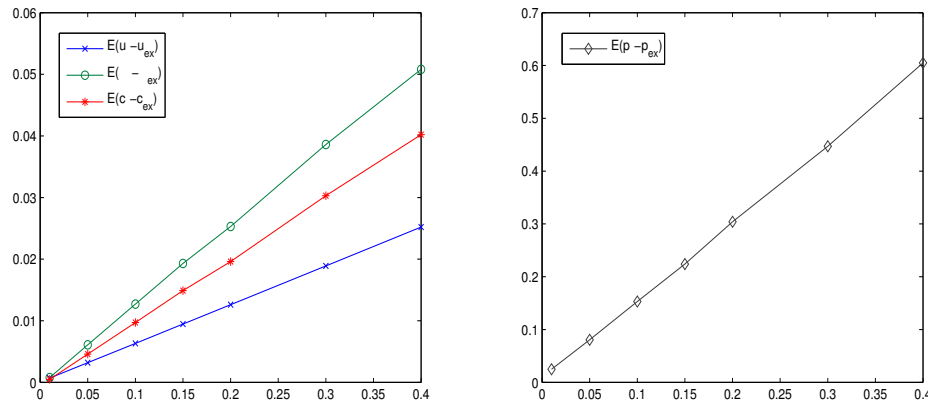


Figure 3.16: Error curves of $E(\Phi_\epsilon - \Phi_{ex})$ versus ϵ .

Slope	$E(\Phi_\epsilon - \Phi_{ex})$	$E(\Phi_\epsilon - \Phi_{app})$
$m_{\mathbf{u}}$	0.0628	0.0635
m_ψ	0.1283	0.1347
m_c	0.1018	0.1065
m_p	1.4877	1.4236

Table 3.15: Slopes of norm $E(\Phi_\epsilon - \Phi_{ex})$ and $E(\Phi_\epsilon - \Phi_{app})$.

pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and concentration $c(\mathbf{x}, t)$. The error curves are given in the Fig. 3.17 for different percentage of the random errors and the slopes of

the error curves are given in the Table 3.15. We again found that the error curves are straight lines with linear dependence of error with respect to ϵ and the slopes of the lines in the first and second cases are approximately same.

Third, the solution curves of velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, phase-field $\psi(\mathbf{x}, t)$ and

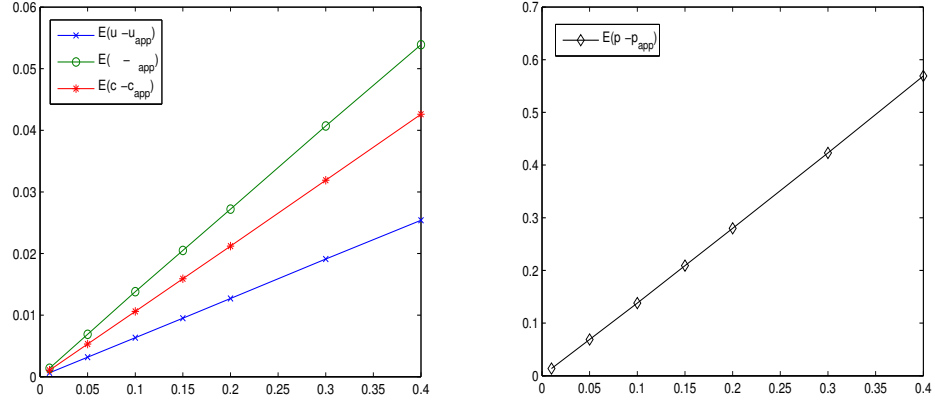
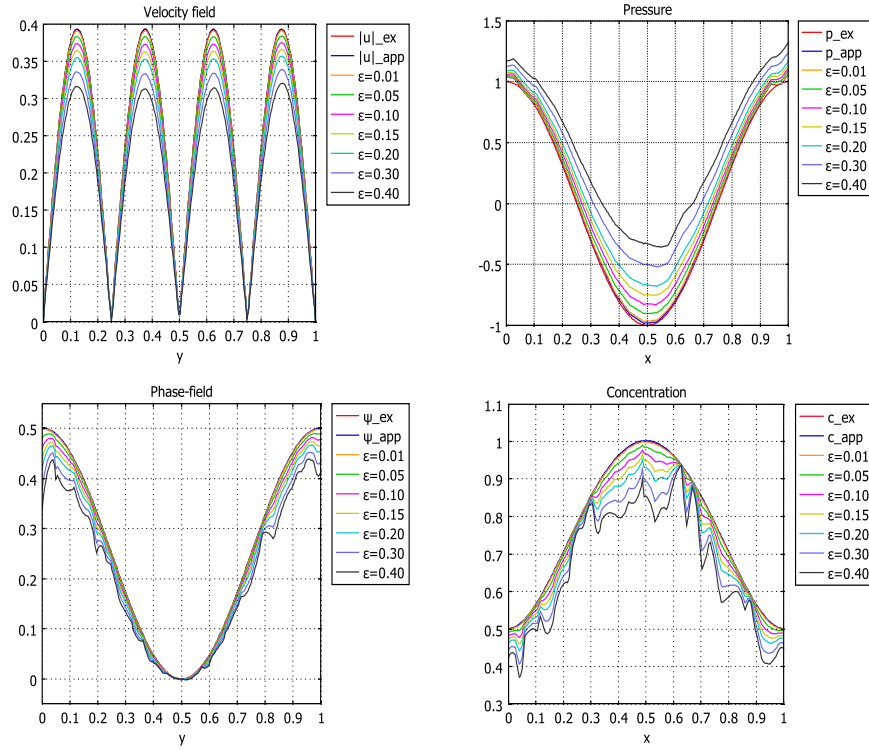


Figure 3.17: Error Curves of $E(\Phi_\epsilon - \Phi_{app})$ versus ϵ .

concentration $c(\mathbf{x}, t)$ for different values of the ϵ are given in the Fig. 3.18. We observe that as we increase the percentage of the random error, i.e. ϵ , the solution is more perturbed, but it does not become unstable.

In Fig. 3.18, we have given the solution of the velocity $\mathbf{u}(\mathbf{x}, t)$ at time $t = 1$, $x = 1/2$ and y varies from 0 to 1, pressure $p(\mathbf{x}, t)$ at time $t = 1$, $y = 1/2$ and x varies from 0 to 1, phase-field $\psi(\mathbf{x}, t)$ at time $t = 1$, $x = 1/2$ and y varies from 0 to 1 and concentration $c(\mathbf{x}, t)$ at time $t = 1$, $y = 1/2$ and x varies from 0 to 1 for the different values of the random error $\epsilon = 0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4$. Also note that the solution corresponding the $\epsilon = 0.00$ is the approximate solution of the model with out random error.

Figure 3.18: Solution curves for the different values of ϵ .

3.5 Conclusion

We have studied the convergence and stability of the numerical scheme (3.19) for the two dimensional isothermal-isotropic and isothermal-anisotropic models (1.103) and (1.101) respectively. We have noticed that in both cases, the numerical scheme is convergent with respect to both spatial and time coordinates and the numerical error estimates are in good accordance with the postulated theoretical error estimates (3.20) and (3.21). Further the developed scheme is stable against the perturbations generated by the inclusion of a random function *randfn* (see Fig. 3.1) in the models. We have noticed that the solution of the models does not become unstable and the error increases linearly as we increase the percentage of the random error in the models. Therefore we can now study the real physical simulations with the real parameters.

Chapter 4

Real Physical Simulations

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4.1 Introduction

In this chapter, we shall present physical simulations of the dendrite growth during the solidification of a Ni-Cu (Nickel-Copper) binary mixture. To perform these simulations, we consider the two dimensional isothermal-anisotropic model (1.97)-(1.100). We further suppose that the interface thicknesses for both substances are equal, *i.e.*, $\delta_A = \delta_B = \delta$, then the problem (1.97)-(1.100) reduces to

$$\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u} + a_1(\psi) \beta_c c(\mathbf{x}, t) \mathbf{G} \\ + a_2(\psi) \sigma_e (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} + \alpha \mathbf{f}(\psi) \quad (4.1a)$$

$$\operatorname{div}(\mathbf{u}) = 0. \quad (4.1b)$$

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla) \psi = M_\psi \epsilon_0^2 \left(\eta^2 \Delta \psi - \frac{\lambda_1(c)}{\delta^2} g'(\psi) - \frac{\lambda_2(c)}{\delta} p'(\psi) \right) \\ - \frac{M_\psi \epsilon_0^2 (\eta \eta'' + (\eta')^2)}{2} \left\{ 2\psi_{xy} \sin 2\theta - \Delta \psi - (\psi_{yy} - \psi_{xx}) \cos 2\theta \right\} \\ + M_\psi \epsilon_0^2 \eta \eta' \left\{ \sin 2\theta (\psi_{yy} - \psi_{xx}) + 2\psi_{xy} \cos 2\theta \right\} \quad (4.1c)$$

$$\frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla) c = \operatorname{div} (D(\psi) \nabla c) + \operatorname{div} \left(\alpha_0 D(\psi) c(1-c) \left(\frac{\lambda'_1(c)}{\delta} g'(\psi) \right. \right. \\ \left. \left. - \lambda'_2(c) p'(\psi) \right) \nabla \psi \right) \quad (4.1d)$$

with the initial and boundary conditions

$$(\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0), \quad \text{in } \Omega. \quad (4.2a)$$

$$\mathbf{u} = 0, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \frac{\partial c}{\partial \mathbf{n}} = 0, \quad \text{on } \Sigma = (0, T_f) \times \partial \Omega. \quad (4.2b)$$

As we know that in the phase-field models the interface thickness δ between the solid-liquid interface must be very small (upto order of 10^{-8} or less), this restriction require a very dense mesh in the simulations of dendrite growth such that the mesh size should be sufficiently less than the interface thickness, otherwise the simulations of dendritic growth cannot be realized. Therefore to perform the simulations of the dendritic growth in the phase-field models, large amount of computational effort or data storage is required. To reduce this requirement, we have constructed two type of meshes for the simulations (for more details see section 4.3). The second kind of mesh contains less number of triangular elements than the first kind of mesh. We have constructed the second kind of mesh in such a way that it does not effect considerably the simulations results but it reduces remarkably the storage requirements and time of execution of the simulations.

First we have nondimensionalized our model (4.1). Second, we have solved a particular case of this model by eliminating the magnetohydrodynamic type system (4.1a)-(4.1b) and compared the results of numerical simulations with the results of the Warren-Boettinger model [24]. Third, to observe the effect of convection only, on dendrite growth, we have included the magnetohydrodynamic type system in our simulations together with the equations of phase-field (4.1c) and concentration (4.1d) by excluding the magnetic-field. Finally we have included the magnetic field and solved the complete set of equations by fixing all other parameters except the magnetic-field. In order to analyze the effect of the magnetic field on the growth of dendrite during the solidification process, we have considered various magnetic fields.

In the next section, we shall provide the non-dimensionalization of the model (4.1).

4.2 Non-dimensionalization of Model Problem

We have non-dimensionalize the model (4.1) by introducing the following dimensionless quantities

$$\begin{aligned}\tilde{\mathbf{x}} &= \frac{\mathbf{x}}{\ell}, & \tilde{t} &= \frac{D_L t}{\ell^2}, & \tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tilde{t}) &= \frac{\ell}{D_L} \mathbf{u}(\mathbf{x}, t), \\ \tilde{\mathbf{B}} &= \frac{\mathbf{B}}{B_0}, & \tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) &= \psi(\mathbf{x}, t), & \tilde{c}(\tilde{\mathbf{x}}, \tilde{t}) &= c(\mathbf{x}, t).\end{aligned}$$

where $\tilde{\mathbf{x}}$ and \tilde{t} are the dimensionless spatial and time coordinates, $\tilde{\mathbf{u}}$, $\tilde{\psi}$, and \tilde{c} are the dimensionless velocity-field, phase-field and concentration respectively, ℓ is the characteristic length of the domain Ω , ℓ^2/D_L is the liquid diffusion time, D_L is the solutal diffusivity in liquid. Note that the phase-field is a mathematical quantity and c is the relative concentration which are already dimensionless quantities. Using these

adimensional relations, we get finally the dimensionless form of the model as

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} = & -\tilde{\nabla} \tilde{p} + Pr \tilde{\Delta} \tilde{\mathbf{u}} + Pr Ra_c a_1(\tilde{\psi}) \tilde{c} \mathbf{e}_G \\ & + Pr (Ha)^2 a_2(\tilde{\psi}) (\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}) \times \tilde{\mathbf{B}} + Kr \mathbf{f}(\tilde{\psi}) \end{aligned} \quad (4.3a)$$

$$\tilde{div}(\tilde{\mathbf{u}}) = 0. \quad (4.3b)$$

$$\begin{aligned} \frac{\partial \tilde{\psi}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\psi} = & \epsilon_2 \left(\eta^2 \tilde{\Delta} \tilde{\psi} - \frac{\lambda_1(\tilde{c})}{\tilde{\delta}^2} g'(\tilde{\psi}) - \frac{\tilde{\lambda}_2(\tilde{c})}{\tilde{\delta}} p'(\tilde{\psi}) \right) \\ & - \frac{\epsilon_2 (\eta \eta'' + (\eta')^2)}{2} \left\{ 2\tilde{\psi}_{xy} \sin 2\theta - \tilde{\Delta} \tilde{\psi} - (\tilde{\psi}_{yy} - \tilde{\psi}_{xx}) \cos 2\theta \right\} \\ & + \epsilon_2 \eta \eta' \left\{ \sin 2\theta (\tilde{\psi}_{yy} - \tilde{\psi}_{xx}) + 2\tilde{\psi}_{xy} \cos 2\theta \right\} \end{aligned} \quad (4.3c)$$

$$\begin{aligned} \frac{\partial \tilde{c}}{\partial \tilde{t}} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{c} = & \tilde{div} \left(\tilde{D}(\tilde{\psi}) \tilde{\nabla} \tilde{c} \right) + \tilde{div} \left(\tilde{\alpha}_0 \tilde{D}(\tilde{\psi}) \tilde{c} (1 - \tilde{c}) \left(\frac{\lambda'_1(\tilde{c})}{\tilde{\delta}} g'(\tilde{\psi}) \right. \right. \\ & \left. \left. - \tilde{\lambda}'_2(\tilde{c}) p'(\tilde{\psi}) \right) \tilde{\nabla} \tilde{\psi} \right) \end{aligned} \quad (4.3d)$$

together with the initial and boundary conditions

$$(\tilde{\mathbf{u}}, \tilde{\psi}, \tilde{c}) (\tilde{t} = 0) = (\tilde{\mathbf{u}}_0, \tilde{\psi}_0, \tilde{c}_0), \quad \text{in } \Omega. \quad (4.4a)$$

$$\tilde{\mathbf{u}} = 0, \quad \frac{\partial \tilde{\psi}}{\partial \mathbf{n}} = 0, \quad \frac{\partial \tilde{c}}{\partial \mathbf{n}} = 0, \quad \text{on } \Sigma = (0, T_f) \times \partial \Omega. \quad (4.4b)$$

where $Pr = \nu/D_L$ is the Prandtl number, $Ra_c = g\beta_c \ell^3/D_L \nu$, is the solutal Rayleigh number, $Ha = (\sigma_e/\rho_0 \nu)^{1/2} B_0 \ell$ is the Hartmann number and $Kr = \alpha \ell^3/\rho_0 D_L^2$, $\tilde{\delta} = \delta/\ell$ is the adimensional interface thickness, $\tilde{\lambda}_2 = \ell \lambda_2$, $\tilde{\alpha}_0 = \alpha_0/\ell$ and $\epsilon_2 = M_\psi \epsilon_0^2/D_L$ are the adimensional parameters. For model parameters, we have used physical values of the binary mixture Ni-Cu as given in the Table 3.1. The density ρ , viscosity μ , and electrical conductivity σ_e are assumed to be constant in the liquid as well as in the solid, therefore we are using average values of Ni and Cu for these constants in the simulations. Also as it is observed experimentally that the dendrites in the Ni-Cu alloy grow with four branches, therefore we have chosen the mode number k in the anisotropic parameter η equal to 4. The adimensional space unit ℓ is chosen as $\ell = 2.8284 \times 10^{-6} m$ which gives the domain length equal to 8 and the domain as $\Omega = [-4, 4] \times [-4, 4]$. With this value of ℓ , we have the adimensional $\tilde{\delta} = 0.03$ which correspond to an interface thickness δ of order $10^{-8} m$. Since the value of δ is strongly dependent on the size of mesh and as the mesh size should be sufficiently less than the interface thickness δ and we are using a coarse mesh for our simulations due to technical difficulties in computations, therefore we fix the value of the adimensional interface thickness as $\tilde{\delta} = 0.05$ for our simulations to ensure the mesh size less than

the interface thickness. The adimensional final time is $t_f = 0.13$, which correspond to the real physical final time of 1 *ms*, with the time step equal to 10^{-5} . Note that big time steps and smaller interface values can create convergence problems during the calculation of numerical solution of the problem.

Initially at the start of solidification, the initial condition is taken to be a circular

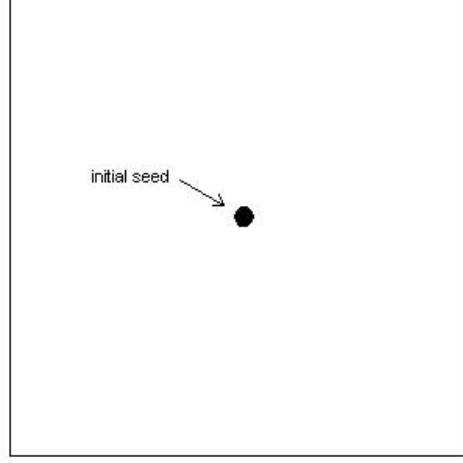


Figure 4.1: Geometry of the problem.

seed of radius 0.2 at the center of the domain Ω . Inside the circular seed the value of ψ is 0 and outside this seed the value of ψ is 1 (see Fig. 4.1). The concentration c in the initial seed is equal to 0.482 and outside the seed it is taken as 0.497, i.e.,

$$\psi(t = 0) = \begin{cases} 0, & x^2 + y^2 < 0.2, \\ 1, & x^2 + y^2 \geq 0.2. \end{cases} \quad (4.5)$$

and

$$c(t = 0) = \begin{cases} 0.482, & x^2 + y^2 < 0.2, \\ 0.497, & x^2 + y^2 \geq 0.2. \end{cases} \quad (4.6)$$

The values of the initial concentration, inside and outside the initial seed, are given different by different authors depending on the phase diagram of binary mixture Ni-Cu (*e.g.*, see [14], [24], [39]).

4.3 Implementation details

This section elaborate the implementation of the model problem (4.3) in COMSOL Multiphysics and MatLab. As described earlier, we have used back-ward Euler's

difference formula for the time discretization and the resulting non-linear fixed point systems are then solved by Newton method (see appendix A for details). For the space discretization we have used mixed finite elements which satisfy the *InfSup* condition (Babuska-Brezis condition) (3.4) for the magnetohydrodynamic type system (4.3a)-(4.3b) and the usual finite elements for phase-field and concentration equations (4.3c) and (4.3d) respectively. In order to solve the obtained linear system, we have used direct method which is the usual Gaussian elimination method. The numerical scheme used to solve the model problem is described in detail in section 3.2. Further in COMSOL Multiphysics, we have used Navier-Stokes equations mode together with no-slip boundary conditions for the magnetohydrodynamic type system, diffusion-convection transient mode together with convective flux boundary conditions for phase-field and concentration equations in two dimensions. However the implementation of our model (4.3) in the COMSOL is not evident. The major difficulty arises in the introduction of anisotropic function and the initial conditions, therefore we have connected the COMSOL with Matlab and used some of the Matlab functions to introduce these functions.

As described earlier that to view the dendrite arms in the simulations of our model, we need a dense mesh. For the coarse mesh, we cannot see the dendrite arms in the simulations. Further, more we refine the mesh, more computer memory is required for the resolution of the model. This makes our model computationally expensive. In order to reduce this difficulty, we have constructed two types of structured meshes, the first type of mesh is uniform everywhere in the domain and generated in a way that first we have divided the domain, at first step, into eight triangles (see Fig. 4.2(a)), at second step each of these eight triangles are further divided into four triangles (see Fig. 4.2(c)), at third step we have divided each triangle further into four triangles and so on. The final mesh used for the simulations is shown in the Fig. 4.2(e) in which there are 128×128 nodes containing 32768 triangular elements.

The second type of mesh is generated in the similar way except that we have made a square given by

$$\Omega_{int} = \{(x, y) \in \mathbb{R}^2 \mid x, y \text{ belongs to square } \mathbb{S}\} \quad (4.7)$$

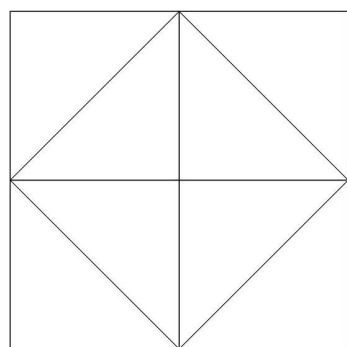
where $\mathbb{S} = \mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_3 \cap \mathbb{L}_4$ such that

$$\begin{aligned} \mathbb{L}_1 &= \{(x, y) \mid y = -x + 4, 0 \leq x \leq 4\}, \\ \mathbb{L}_2 &= \{(x, y) \mid y = x + 4, 0 \leq x \leq -4\}, \\ \mathbb{L}_3 &= \{(x, y) \mid y = -x - 4, 0 \leq x \leq -4\}, \\ \mathbb{L}_4 &= \{(x, y) \mid y = x - 4, 0 \leq x \leq 4\}. \end{aligned}$$

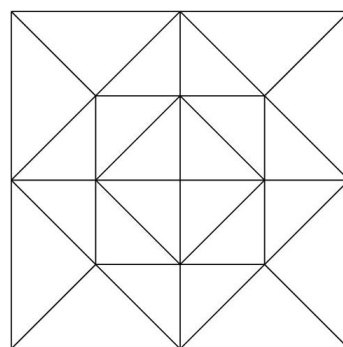
inside the domain $\Omega = [-4, 4] \times [-4, 4]$ and the triangles inside Ω_{int} are divided two times greater than the triangles outside Ω_{int} (see Fig. 4.2). The final mesh used in simulations has 128×128 nodes inside and 64×64 nodes outside the square Ω_{int} containing 24576 triangular elements (see Fig. 4.2(f)). The second kind of mesh is used to save the computational time and to reduce memory requirements without having effect on the results.

We have used two types of mixed finite elements to solve the problem (4.3). First is $\mathbb{P}_2 - \mathbb{P}_1$ for the magnetohydrodynamic type system and \mathbb{P}_2 finite elements for the phase-field and concentration equations respectively. Second is the $\mathbb{P}_3 - \mathbb{P}_2$ for the magnetohydrodynamic type system and \mathbb{P}_3 for the phase-field and concentration equations of the problem (4.3). The adimensional time step is fixed as 10^{-5} with the final time equal to $t_f = 0.13$ in each simulation unless otherwise mentioned. It is important to mention that using $(\mathbb{P}_2 - \mathbb{P}_1)$ for the final time 0.13 and type-I mesh, it takes approximately 29 hours and using type-II mesh takes approximately 18 hours to complete one simulation. And using $(\mathbb{P}_3 - \mathbb{P}_2)$ with type-II mesh, it takes about 8 days to execute one simulation using the hardware defined below.

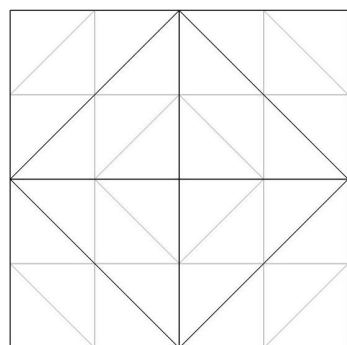
To carry out all simulations we have used a Dell Laptop computer with 4GB of computer memory and 2GHz core² dual processor with 64-bit Vista windows.



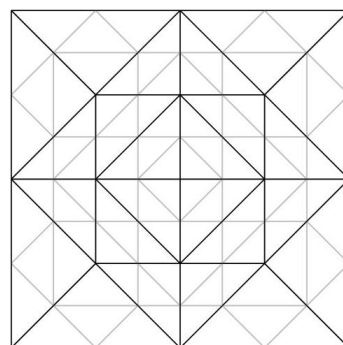
(a) Mesh type-I at first step.



(b) Mesh type-II at first step.



(c) Mesh type-I at second step.



(d) Mesh type-II at second step.

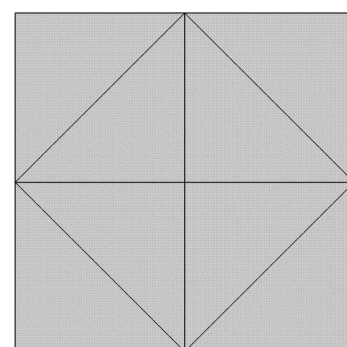
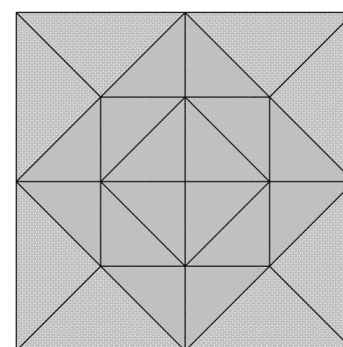
(e) Mesh type-I at final step with 128×128 nodes.(f) Mesh type-II at final step with 64×64 nodes outside and 128×128 nodes inside Ω_{int} .

Figure 4.2: Types of mesh used in simulations.

4.4 Physical Simulations

In this section, we shall present simulations of our model problem (4.3) for different cases. First, we shall solve the model for a particular case in which we eliminate the magnetohydrodynamic type system (4.3a)-(4.3b), *i.e.*, we consider only phase-field and concentration equations (4.3c) and (4.3d) to execute the simulation. Second, we shall present the simulations of our model by including the magnetohydrodynamic type system and present the results obtained by introducing different magnetic fields.

4.4.1 Reduced Model (Warren-Boettinger type Model)

In the model problem (4.3), if we assume that there is no motion in the melt during the solidification process, then the magnetohydrodynamic type system will be eliminated from the model (4.3). Consequently the convection terms $(\mathbf{u} \cdot \nabla) \psi$ and $(\mathbf{u} \cdot \nabla) c$ in the phase-field and concentration equations (4.3c) and (4.3d) will also be eliminated and these equations will become simple diffusion equations of Warren-Boettinger type model [24]. Then we have solved these equations using \mathbb{P}_2 finite elements and the type-I mesh for 210053 degree of freedom. The plots of phase-field and concentration and their contour plots are presented in the Fig. 4.3 and 4.4 respectively. We can see that the dendrites obtained in this case are completely symmetric about x and $y - axis$ as expected in the simulations of the Warren-Boettinger model. It is to be noted that, we have not obtained exactly same form of the dendrites as in Warren-Boettinger model because we have used coarse mesh as compared to the mesh used in the Warren-Boettinger simulations.

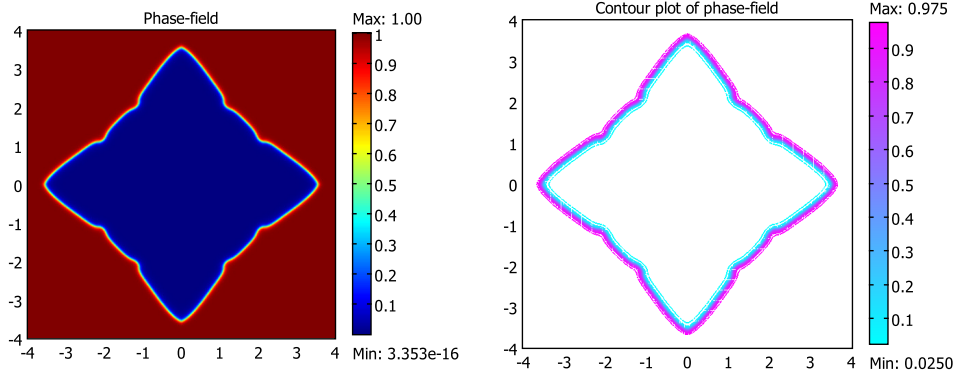


Figure 4.3: Plots of phase-field variable.

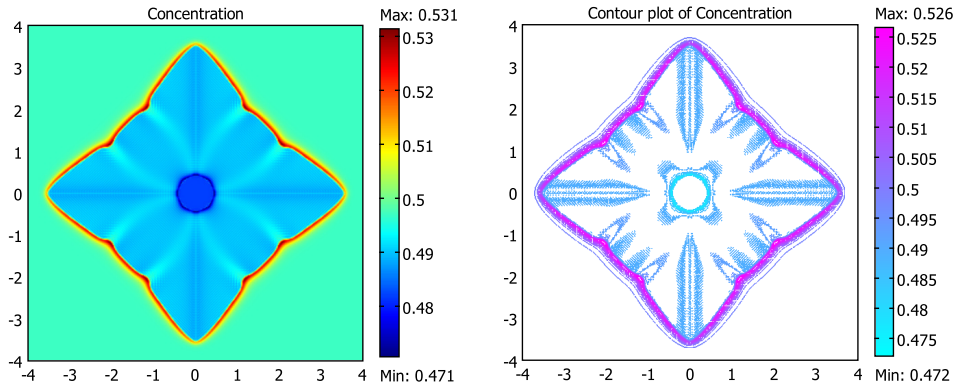


Figure 4.4: Plots of concentration.

4.4.2 Our Model

We incorporate here magnetohydrodynamic type system in the simulation of dendrite growth and consider the complete set of the model equations (4.3). To investigate the effect of convection on the dendrite growth, we have considered, first, the model (4.3) with out magnetic field *i.e.*, $\tilde{\mathbf{B}} = 0$ and by assuming $Kr = 0$. The velocity-field, phase-field and concentration and their contour plots for the first case, are given in Fig. 4.5, 4.6 and 4.7 respectively. We observe from results in this case that the magnitude of

velocity is very small, therefore there is no significant change in the dendrite structure due to the convection during the solidification process and it remains symmetric as it was in the case of Warren-Boettinger type model. Note that in Fig. 4.5 and all subsequent figures of velocity field, we have presented the plots of velocity times phase field (i.e., $\mathbf{u} \times \psi$) to show the velocity around the dendrite.

Second, to observe the effect of magnetic field on the dendrite growth, we have fixed all other parameters and solved the problem (4.3) by varying magnetic field at angles 45° , 90° and a variable magnetic field. All these simulations are performed, first by using \mathbb{P}_2 finite elements for the velocity, phase-field and concentration and \mathbb{P}_1 for the pressure. Type-I mesh, with 128×128 nodes, is used to solve the problem in all cases except in the case where the magnetic field is applied to 90° . In this case, the problem is solved using mesh type-II just to show that using second kind of mesh does not effect the results considerably. Second we have executed these simulations for the magnetic field applied at an angle 45° and variable magnetic field, using \mathbb{P}_3 finite elements for the velocity, phase-field and concentration and \mathbb{P}_2 for the pressure and type-II mesh.

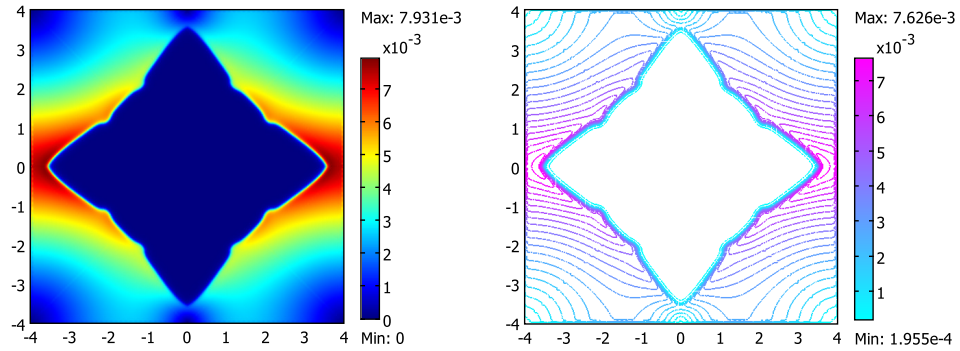


Figure 4.5: Plots of velocity-field.

4.4.2.1 Magnetic-Field at an Angle 45°

In Fig. 4.8, 4.9 and 4.10, the plots of velocity-field, phase-field and concentration and their contour plots are presented by introducing magnetic field at an angle of 45° , (i.e., $\tilde{\mathbf{B}} = 1/\sqrt{2}(1, 1)$) to solve the model problem (4.3) using type-I mesh and \mathbb{P}_2 finite elements for the velocity, phase-field and concentration and \mathbb{P}_1 for the pressure.

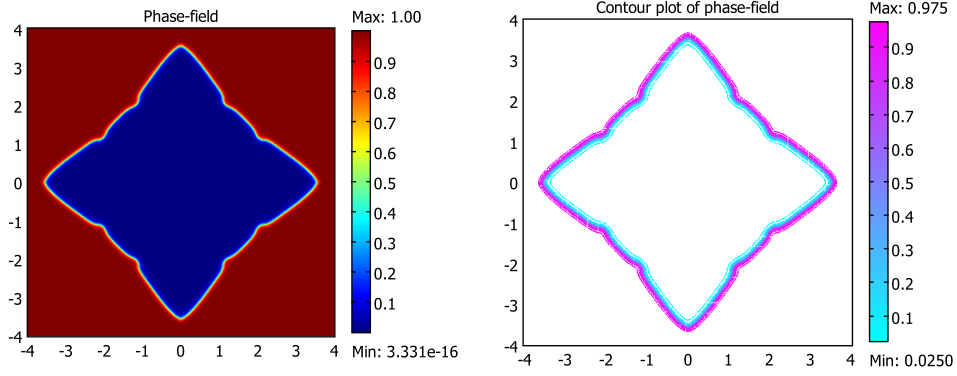


Figure 4.6: Plots of phase-field variable.

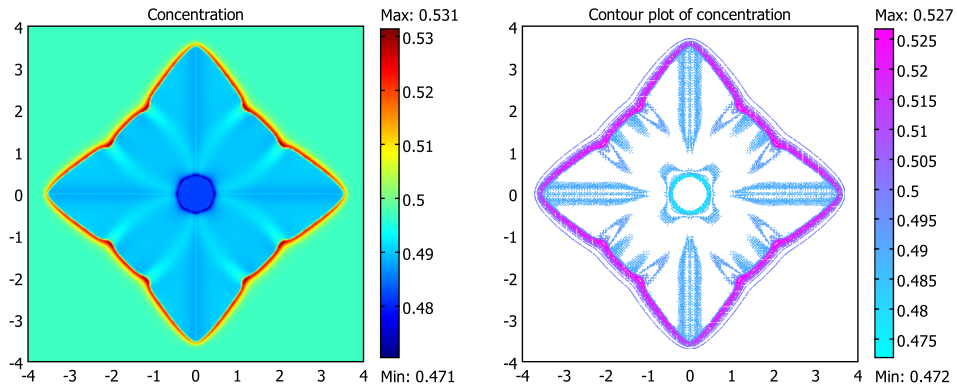


Figure 4.7: Plots of concentration.

We have observed that by the introduction of magnetic field, the magnitude of velocity has increased and the dendrite tips grow more rapidly and collide with the boundary of the domain and it is no more symmetric about x and y – axis while in the previous cases the dendrite tips are far from the boundary with the same number of iterations at the final adimensional time $t = 0.13$. The form of the dendrite has been changed significantly and it is now symmetric about the line $y = x$. This change in the form of the dendrite is due to the introduction of magnetic field at an angle 45° .

Further we have solved the same model using \mathbb{P}_3 finite elements for the velocity, phase-field and concentration and \mathbb{P}_2 for the pressure using type-II mesh and presented the

results in Fig. 4.18. We observe in this case that the dendrite is more refined and the secondary dendrite arms have also been started growing up along the primary dendrite arms. We can also see the effect of magnetic field more clearly as the top and right dendrite arms are smaller than the bottom and left dendrite arms. Also in this case the dendrite arms are far from the boundary of the domain whereas in the case where the finite elements were \mathbb{P}_2 for the velocity, phase-field and concentration and \mathbb{P}_1 for the pressure, the dendrite arms collide with the boundary of the domain.

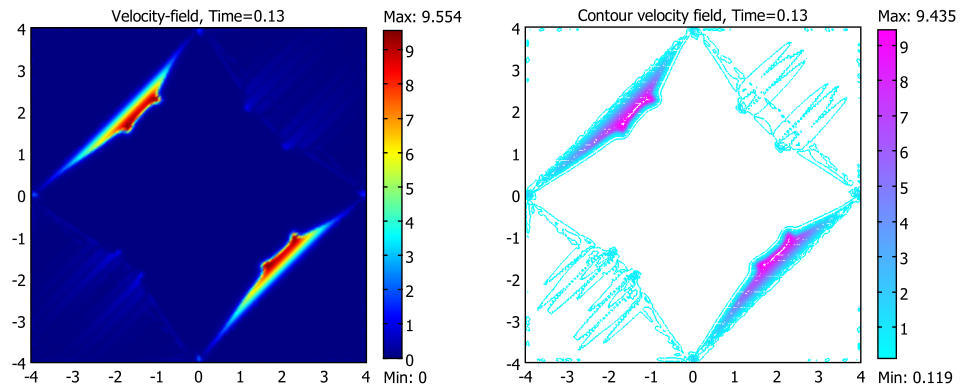


Figure 4.8: Plots of velocity-field.

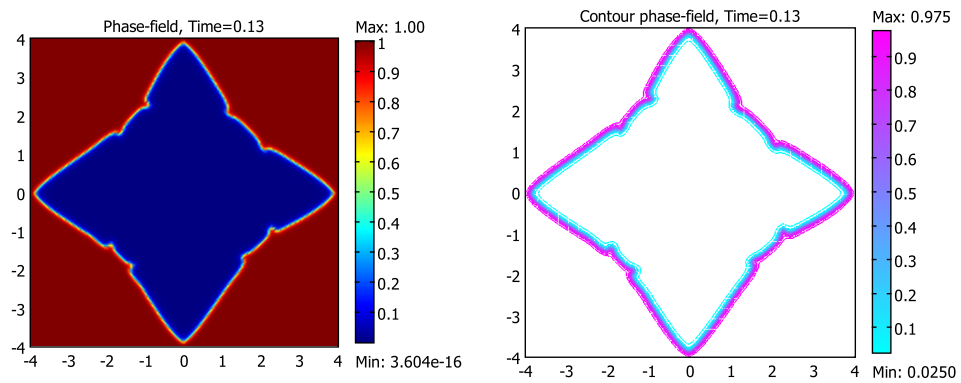


Figure 4.9: Plots of phase-field variable.

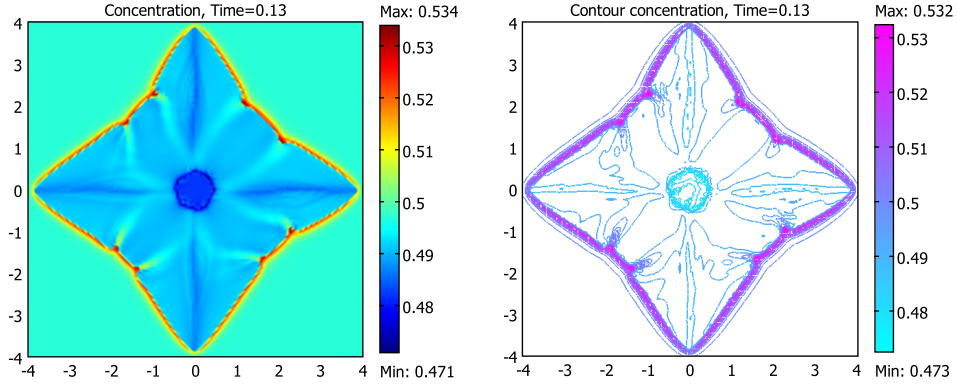


Figure 4.10: Plots of concentration.

4.4.2.2 Magnetic-Field at an Angle 90°

Next we have solved the model problem (4.3) by applying the magnetic field at an angle of 90° (i.e., $\tilde{\mathbf{B}} = (0, 1)$) and presented the results in Fig. 4.11, 4.12 and 4.13. We observe that the form of dendrite has not been changed considerably, but the dendritic arms along x -axis are little longer than the arms along the y -axis and it is now symmetric only about the y -axis. We have verified this behavior of the dendrite arms by applying the magnetic field at 0° and found that in this case the dendrite arms along y -axis have grown up little larger than the dendrite arms along x -axis. In this case our results are in good accordance with the observation of, for example [48], who examined that the constant magnetic-field does not effect significantly the inter-dendritic flows and micro-segregation during the solidification process.

4.4.2.3 Variable Magnetic-Field

In this case, we have introduce a variable magnetic field $\tilde{\mathbf{B}} = (\cos(x), \sin(y))$ in the model (4.3) and obtained the simulations using type-I mesh for \mathbb{P}_2 finite elements for the velocity, phase-field and concentration and \mathbb{P}_1 for the pressure. The plots of velocity-field, phase-field and concentration are given in Fig. 4.14, 4.15 and 4.16. We found an irregular structure of dendrite in this case and notice that the magnitude of velocity has also been increased greatly. We can see that the dendrite is no more symmetric about any axis and it is deformed drastically by the introduction of variable magnetic field. The left arm of the dendrite has grown up more than the right arm and they have completely different shape from each other.

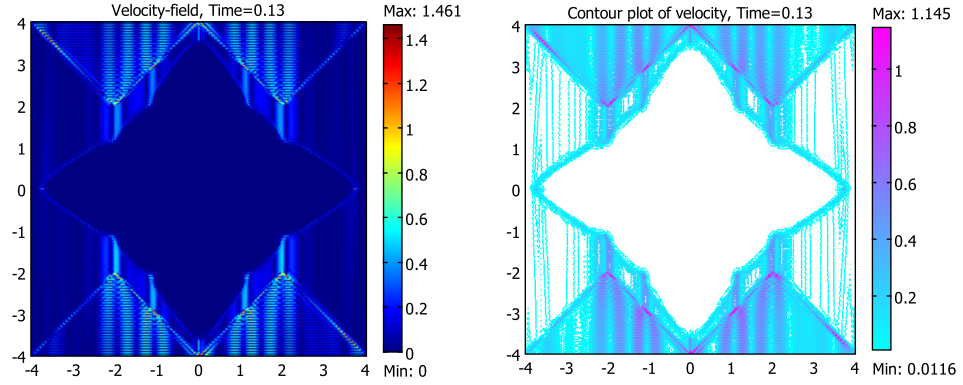


Figure 4.11: Plots of velocity-field.

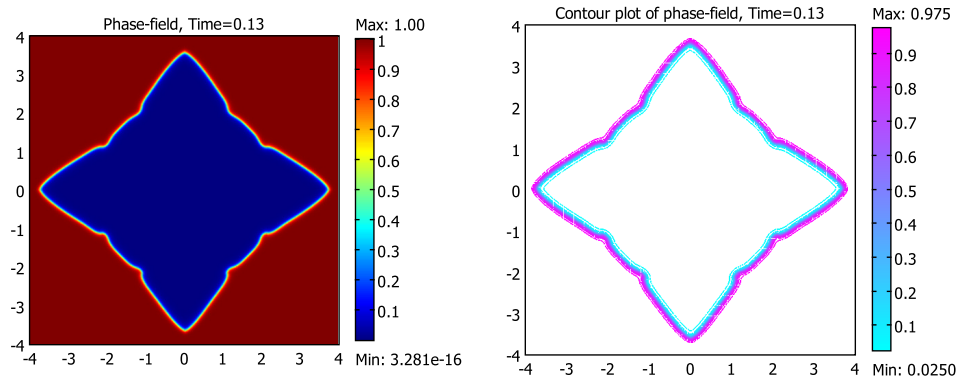


Figure 4.12: Plots of phase-field variable.

Further we have solved the same problem using \mathbb{P}_3 finite elements for the velocity, phase-field and concentration and \mathbb{P}_2 for the pressure using type-II mesh and give the results in Fig. 4.19. We can see that dendrite structure in this case has been changed significantly, large secondary arms arise along the left arm of the dendrite whereas along the other arms of the dendrite, the secondary arms have not grown up greatly.

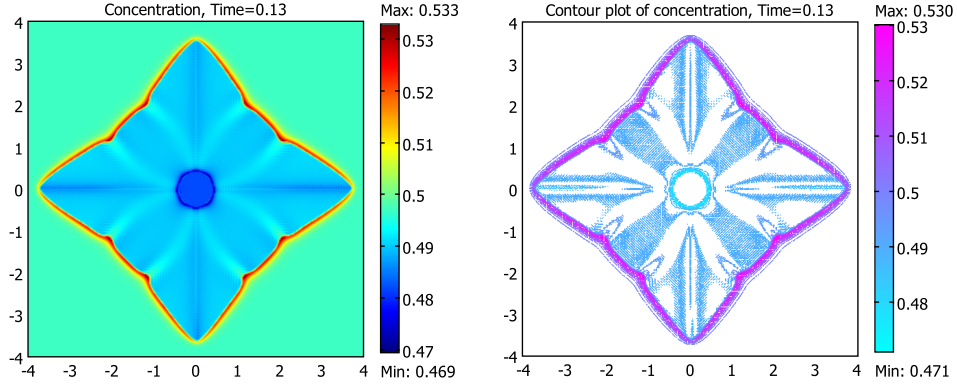


Figure 4.13: Plots of concentration.

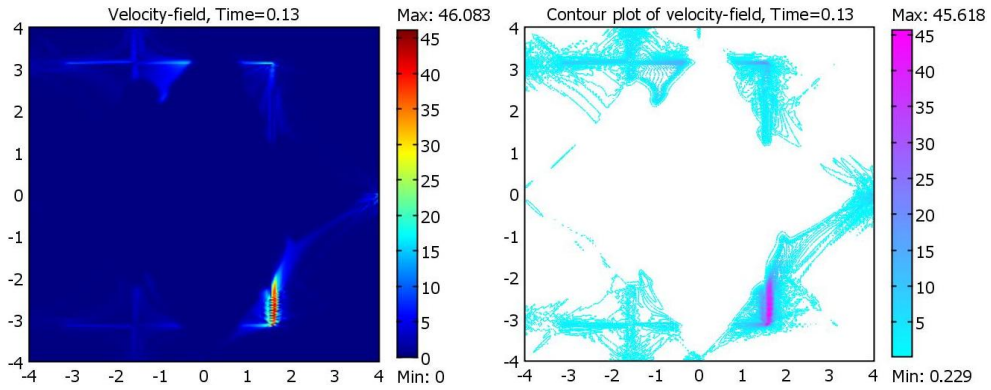


Figure 4.14: Plots of velocity-field.

4.4.3 Dendrite Comparison

Now we shall present comparison between the dendrites obtained by solving the Warren-Boettinger type model and our model (4.3) with different magnetic-fields to show the effect of magnetic-field on the structure of dendrite. For this we consider only the simulations results obtained using \mathbb{P}_2 finite elements for the velocity, phase-field and concentration and \mathbb{P}_1 for the pressure. Also we consider only the contour plots of phase-field at $\psi = 0.5$ and final time $t_f = 0.13$. In each of Fig. 4.17, we have shown the plots of phase field for the Warren-Boettinger type model (WBTM) and our model with magnetic-field at angles 45° , 90° and variable magnetic-field respectively. In Fig. 4.17(a), we notice that the form of the dendrite for the WBTM is symmetric about x and y – axis and by the application of magnetic-field at an angle

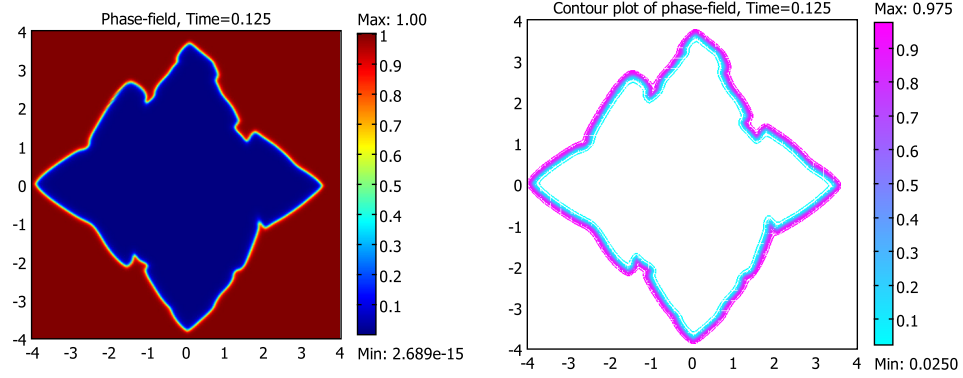


Figure 4.15: Plots of phase-field.

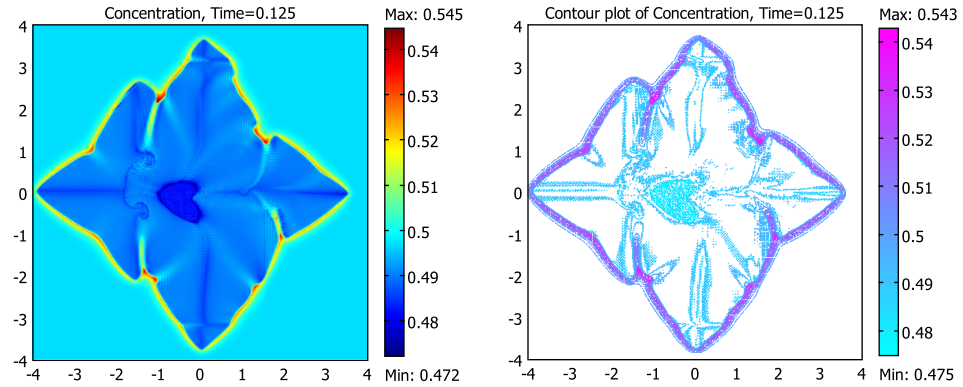


Figure 4.16: Plots of concentration.

45° , the form of the dendrite has been deformed, it is no longer symmetric along x and y – axis, the top and right dendrite arms are smaller than the bottom and left dendrite arms and it is symmetric about the line $y = x$. In Fig. 4.17(b), we can see that by the introduction of magnetic-field at an angle 90° , the dendrite has grown up little longer along x – axis. And in Fig. 4.17(c), we observe that by applying variable magnetic-field, the form of dendrite is completely deformed and it is no longer symmetric as it was in the WBTM.

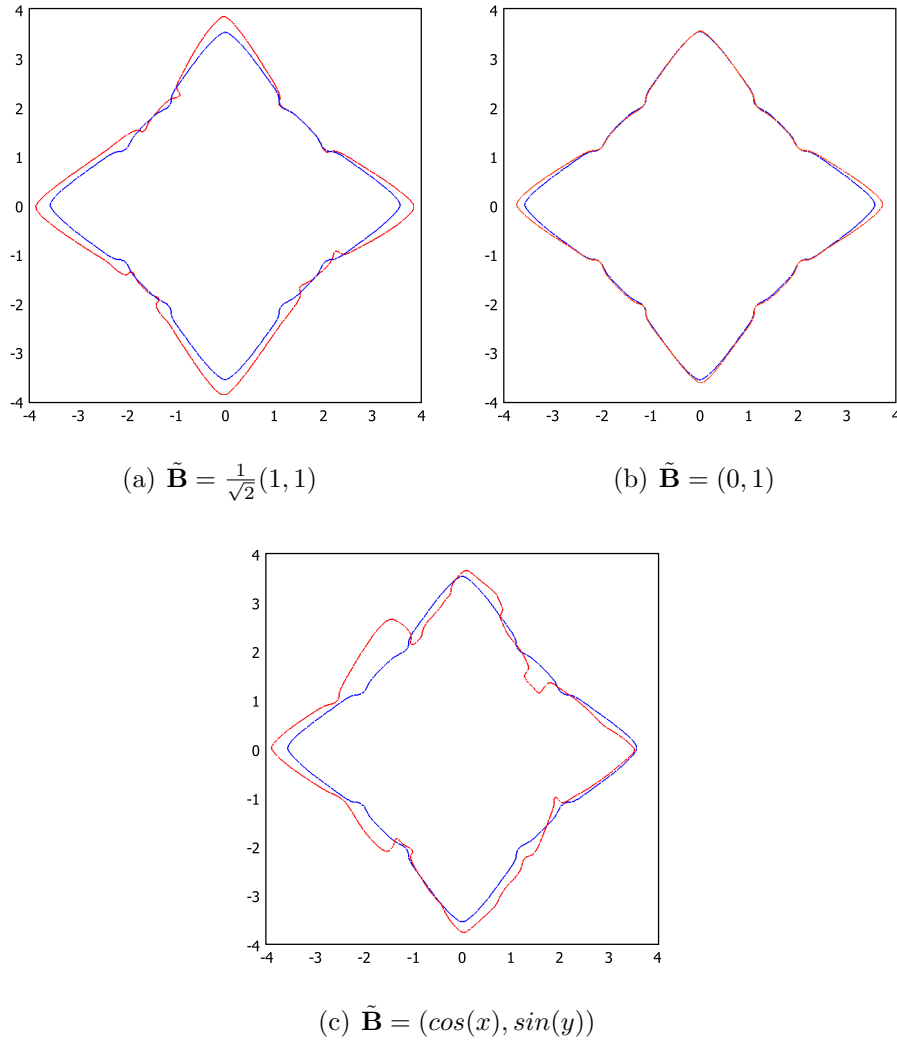


Figure 4.17: Comparison of dendrite obtained in WBTM and our model for different magnetic fields.

4.5 Conclusion

We have presented the realistic numerical simulations for the anisotropic solidification of binary alloys by considering an example of Ni-Cu mixture with real physical parameters in this chapter. The simulations have been carried out by using two type of structured meshes. The second type of mesh reduced the computational time and storage requirements noticeably without having effect on the results. Also two types of (mixed) finite elements are used in these simulations. Our main focus in these simulations was the influence of magnetic-field on the structure of dendritic growth during the solidification process. We have considered various types of magnetic-fields to show

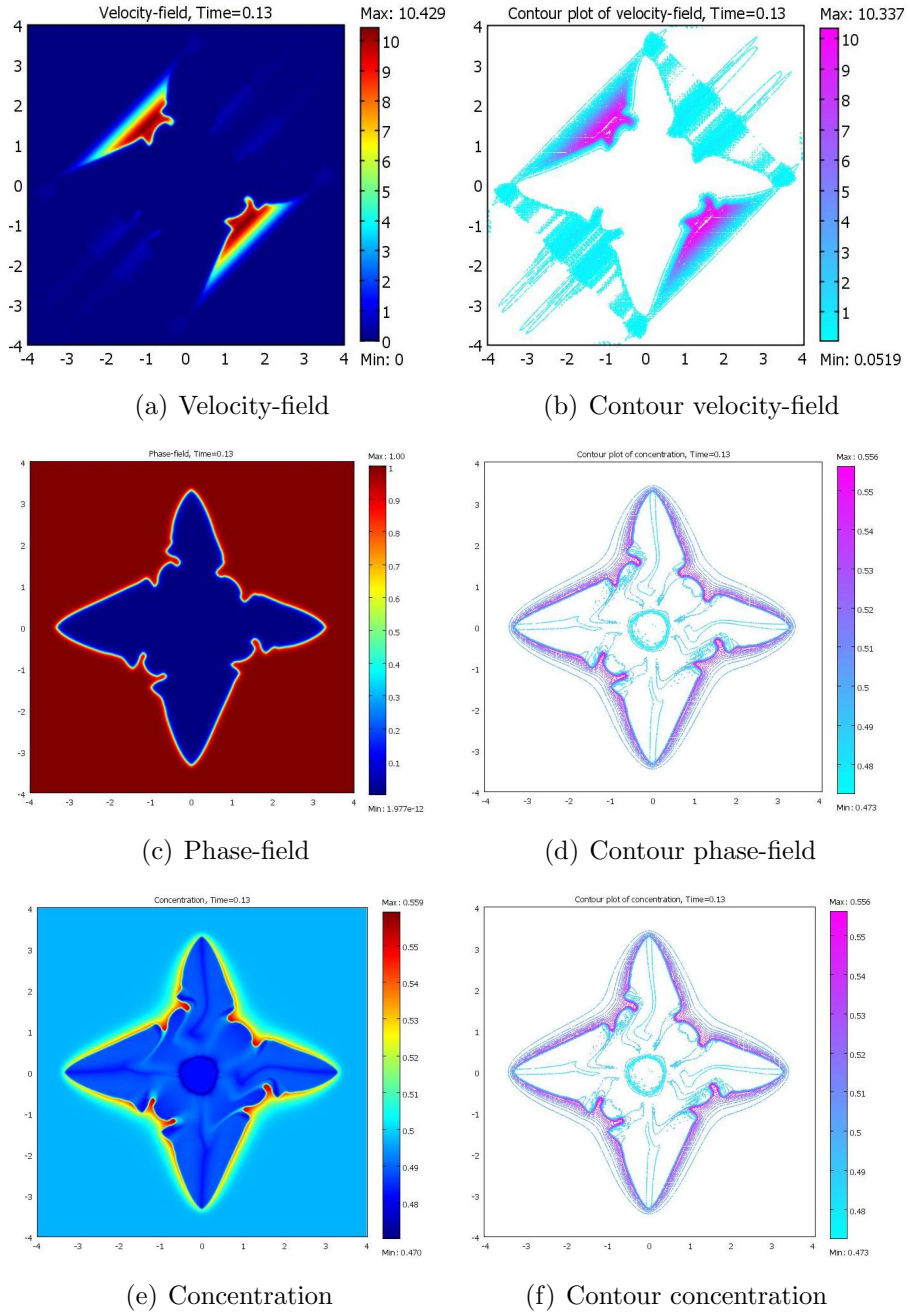


Figure 4.18: Plots of velocity, phase-field and concentration obtained for variable magnetic field $\tilde{\mathbf{B}} = \frac{1}{\sqrt{2}}(1, 1)$ using type-II mesh and \mathbb{P}_3 for the velocity, phase-field and concentration and \mathbb{P}_2 for the pressure.

the influence on the dendrites. We have observed that the constant magnetic-field do not have significant impact on the dendrites whereas the variable magnetic-field have

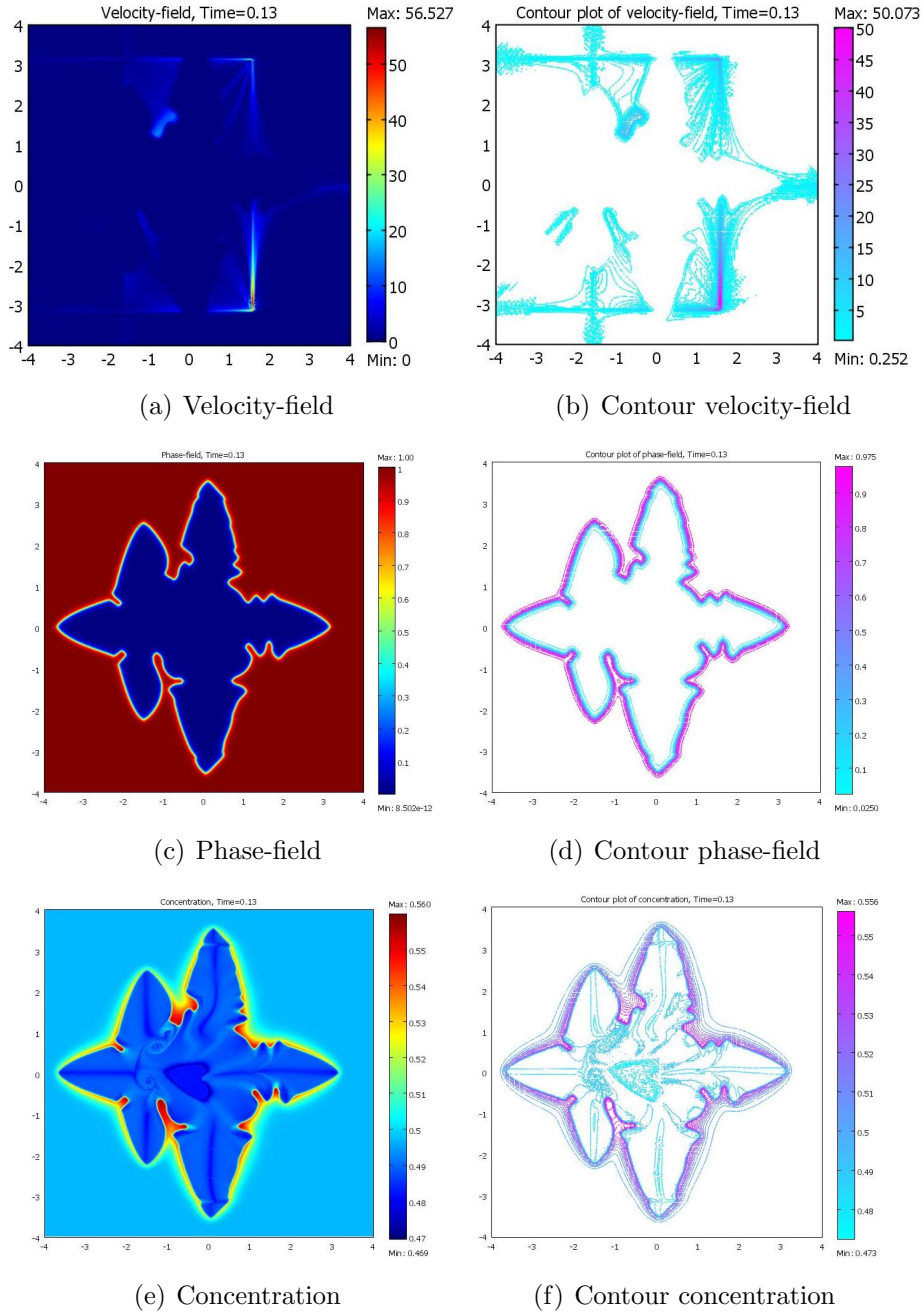


Figure 4.19: Plots of velocity, phase-field and concentration obtained for variable magnetic field $\tilde{\mathbf{B}} = (\cos(x), \sin(y))$ using type-II mesh and \mathbb{P}_3 for the velocity, phase-field and concentration and \mathbb{P}_2 for the pressure.

deformed the dendrites considerably. Our results agree to the observations made by [48].

Chapter 5

Control Problem

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5.1 Introduction

The aim of this chapter is to study the optimal control of our model that governs the solidification process of a binary alloy in the presence of motion in the liquid phase with the magnetic field effect in an isothermal environment. As we have seen in the realistic numerical simulations (see chapter 4) that the application of magnetic field has a considerable effect on the evolution of dendrites and we can control the direction of motion of the melt during the solidification process. Our main focus in this study is to control, by the action of magnetic-field, the desired dynamics of the melt. Therefore we shall formulate the optimal control by considering the magnetic field as a control variable. The cost function measures the distance between the calculated and desired dynamics.

To study the optimal control, we have considered an adimensionalized problem (4.3) given in the chapter 4. In order to take into account the influence of temperature on the magnetic-field, we decompose the operator $\mathbf{f}(\psi)$ in the problem (4.3) as $\mathbf{f}(\psi) = Kr\tilde{\mathbf{f}}(\psi) + \kappa a_1(\psi)\mathbf{B}$, where the second term in this operator correspond to the fluctuations of solidification temperature on the magnetic-field. Further we have reduced this model for the isotropic case, that is, the anisotropy η is assumed to be constant, then we will have the following primal problem.

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + \alpha \Delta \mathbf{u} + \mathcal{A}_1(\psi, c) \\ &\quad + b(\psi)(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} + \kappa a_1(\psi)\mathbf{B}, \quad \text{on } \mathcal{Q}, \end{aligned} \quad (5.1a)$$

$$\operatorname{div}(\mathbf{u}) = 0, \quad \text{on } \mathcal{Q}, \quad (5.1b)$$

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla)\psi = \epsilon_2 \Delta \psi - \mathcal{A}_2(\psi, c), \quad \text{on } \mathcal{Q}, \quad (5.1c)$$

$$\frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla)c = \operatorname{div}(D(\psi)\nabla c) + \operatorname{div}(\mathcal{A}_3(\psi, c)\nabla \psi), \quad \text{on } \mathcal{Q}, \quad (5.1d)$$

$$(\mathbf{u}, \psi, c)(t=0) = (\mathbf{u}_0, \psi_0, c_0), \quad \text{in } \Omega, \quad (5.1e)$$

$$\mathbf{u} = \mathbf{0}, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \frac{\partial c}{\partial \mathbf{n}} = 0, \quad \text{on } \Sigma. \quad (5.1f)$$

where in this section $\alpha = Pr$, $b(\psi) = Pr(Ha)^2 a_2(\psi)$ and

$$\mathcal{A}_1(\psi, c) = PrRa_c a_1(\psi) c \mathbf{e}_G + Kr\tilde{\mathbf{f}}(\psi), \quad (5.2)$$

$$\mathcal{A}_2(\psi, c) = \frac{\lambda_1(c)}{\delta^2} g'(\psi) + \frac{\lambda_2(c)}{\delta} p'(\psi), \quad (5.3)$$

$$\mathcal{A}_3(\psi, c) = \alpha_0 D(\psi) c (1 - c) \left(\frac{\lambda'_1(c)}{\delta} g'(\psi) - \lambda'_2(c) p'(\psi) \right). \quad (5.4)$$

We assume in the sequel that initial conditions satisfy the regularity:

$$(\mathbf{u}_0, \psi_0, c_0) \in (H_0^1(\Omega))^2 \times H_0^2(\Omega) \times H^1(\Omega).$$

Then according to the Theorems 1 and 4, the problem (5.1) admits a unique solution $\mathbf{X} = (\mathbf{u}, \psi, c) \in \mathcal{W} = \mathcal{W}_1^1 \times \mathcal{W}_2^1 \times \mathcal{W}_1^1$, where \mathcal{W}_i^1 , $i=1,2$, are defined in section 2.6.

The control problems related to the phase-field models have been studied by [36], in which the authors examined the optimal control of the solidification of pure materials due to thermal effects. Robust control and stabilization problems associated with solidification of pure materials due to thermal effects and isothermal Warren-Boettinger type model of binary mixtures in order to take into account the influence of noises in the data have been studied by [6]-[8]. For the control problems using magnetic field as a control variable, we can cite [38], in which the authors studied the optimal strategy for the suppression of the turbulent motions in melt flows and [8] has discussed the defects by stabilizing the melt flow motion during the growth process.

In the section 5.2, we shall present the formulation of the control problem and discuss the existence of the optimal solutions. In the subsequent section (5.3), the optimality conditions as well as the adjoint problem corresponding to the primal problem are given.

5.2 Optimal Control Problem

In this section, we shall formulate the control problem and define the essential spaces used in the optimal control. Further we shall define the cost functional and the give the existences results.

5.2.1 Formulation of the problem and existence result

Let \mathcal{U} be a space of controls which is assumed to be Hilbert space and let \mathcal{U}_a be its non-empty closed subset which is taken as admissibility set of controls and is defined by

$$\mathcal{U}_a = \left\{ \mathbf{B} = (B_1, B_2) \in (L^2(\Omega))^2 \mid 0 < \mathbf{b}_1 \leq B_1 \leq \mathbf{b}_2 < \infty, \right. \\ \left. 0 < \mathbf{b}'_1 \leq B_2 \leq \mathbf{b}'_2 < \infty \right\}. \quad (5.5)$$

The optimal control, we consider is the following:

Find $(\mathbf{X}, \mathbf{B}) \in \mathcal{W} \times \mathcal{U}_a$ such that the following cost functional

$$J(\mathbf{B}) = \frac{\alpha_1}{2} \|\mathbf{u} - \mathbf{u}_{obs}\|_{L^2(\mathcal{Q})}^2 + \frac{\alpha_2}{2} \|\psi - \psi_{obs}\|_{L^2(\mathcal{Q})}^2 + \frac{\alpha_3}{2} \|c - c_{obs}\|_{L^2(\mathcal{Q})}^2 + \frac{\beta}{2} \|\mathbf{B}\|_{L^2(\mathcal{Q})}^2, \quad (5.6)$$

is minimized subject to the problem (5.1),

where $\sum_{i=1}^3 \alpha_i > 0$ and $\alpha_i \geq 0$ for $i = 1, 2, 3$, and $\mathbf{X}_{obs} = (\mathbf{u}_{obs}, \psi_{obs}, c_{obs}) \in L^2(\mathcal{Q})$ is given and represent the target variable.

More precisely the problem is to find an optimal solution $\mathbf{B} \in \mathcal{U}_a$ such that

$$J(\mathbf{B}^*) = \inf_{\mathbf{B} \in \mathcal{U}_a} J(\mathbf{B}). \quad (5.7)$$

and \mathbf{X}^* is the solution of the problem (5.1), corresponding to \mathbf{B} .

Further to study the optimal control of the problem (5.1) we need more assumptions on the operators. Therefore in addition to the hypothesis (H1)-(H5), we suppose that the operators $\mathcal{A}_1(\psi, c)$, $\mathcal{A}_2(\psi, c)$, $\mathcal{A}_3(\psi, c)$, $D(\psi)$, $b(\psi)$ and $a_1(\psi)$ in the problem (5.1) satisfy the following hypothesis

(H6) \mathcal{A}_1 is differentiable with respect to ψ and c and its derivative is Lipschitz continuous a.e. in \mathcal{Q} .

(H7) \mathcal{A}_i $i = 2, 3$ are differentiable with respect to ψ and c and their derivatives are Lipschitz continuous a.e. in \mathcal{Q} .

(H8) D , b and a_1 are differentiable with respect to ψ and their derivatives are Lipschitz continuous a.e. in \mathcal{Q} .

Now we shall give the existence result of the problem (5.7).

Theorem 5 *Under the assumptions of Theorems 1 and 4, the optimal control (5.7) has at least one solution \mathbf{B}^* in \mathcal{U}_a .*

Proof: It is easy to verify that $\min J(\mathbf{B})$ is finite in \mathcal{U}_a , and thus, there exists a minimizing sequence \mathbf{B}_n in \mathcal{U}_a such that

$$\lim_{n \rightarrow \infty} J(\mathbf{B}_n) = \inf_{\mathbf{B} \in \mathcal{U}_a} J(\mathbf{B}).$$

This implies the uniform boundedness of \mathbf{B}_n in $L^2(\mathcal{Q})$ and therefore there is a subsequence, denoted also by, \mathbf{B}_n such that

$$\mathbf{B}_n \rightharpoonup \mathbf{B}^* \text{ weakly in } L^2(\mathcal{Q}).$$

Similar to the proof of Proposition 11.16 in Belmiloudi [8], we can show that the operator \mathcal{F} is continuous from the weak topology of $L^2(\mathcal{Q})$ to the strong topology of $(L^2(\mathcal{Q}))^4$. Consequently

$$\mathcal{F}(\mathbf{B}_n) \rightarrow \mathcal{F}(\mathbf{B}^*) \text{ strongly in } (L^2(\mathcal{Q}))^4.$$

Since J is weakly lower semi-continuous, we have

$$J(\mathbf{B}^*) \leq \liminf_{n \rightarrow \infty} J(\mathbf{B}_n).$$

and then

$$J(\mathbf{B}^*) = \inf_{\mathbf{B} \in \mathcal{U}_a} J(\mathbf{B}).$$

which achieves the proof. \square

In the next section we shall discuss the differentiability and present optimality conditions.

5.3 Optimality Conditions

We introduce now the following mapping: $\mathcal{F} : \mathcal{U}_a \rightarrow \mathcal{W}$, which maps the source term $\mathbf{B} \in \mathcal{U}_a$ into the corresponding solution \mathbf{X} of (5.1) in \mathcal{W} .

Before proceeding to the optimality conditions of the optimal solution, we shall give the G-differentiability of the operator \mathcal{F} .

Theorem 6 *Under the assumptions of Theorems 1 and 4 and the hypothesis (H6)-(H8), the function \mathcal{F} is G-differentiable with respect to \mathbf{B} , where its G-derivative*

$$\mathcal{F}'(\mathbf{B}) : \mathcal{B} \rightarrow \mathbf{Y} = \mathcal{F}'(\mathbf{B}) \cdot \mathcal{B},$$

with $\mathbf{Y} = (w_1, w_2, w_3)$, is the unique solution in \mathcal{W} of the following linear problem

$$\begin{aligned}
& \frac{\partial \mathbf{w}_1}{\partial t} + (\mathbf{w}_1 \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{w}_1 = -\nabla \pi + \alpha \Delta \mathbf{w}_1 \\
& \quad + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial \psi} \cdot w_2 + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial c} \cdot w_3 + b'(\psi) w_2 ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \\
& \quad + b(\psi) \left\{ (\mathbf{w}_1 \times \mathbf{B}) \times \mathbf{B} + (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} + (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \right\} \\
& \quad + \kappa a'_1(\psi) \mathbf{B} w_2 + \kappa a_1(\psi) \mathbf{B}, \quad \text{on } \mathcal{Q} \\
& \operatorname{div}(\mathbf{w}_1) = 0, \quad \text{on } \mathcal{Q} \\
& \frac{\partial w_2}{\partial t} + (\mathbf{w}_1 \cdot \nabla) \psi + (\mathbf{u} \cdot \nabla) w_2 = \epsilon_2 \Delta w_2 - \frac{\partial \mathcal{A}_2(\psi, c)}{\partial \psi} w_2 - \frac{\partial \mathcal{A}_2(\psi, c)}{\partial c} w_3, \\
& \frac{\partial w_3}{\partial t} + (\mathbf{w}_1 \cdot \nabla) c + (\mathbf{u} \cdot \nabla) w_3 = \operatorname{div} \left(D'(\psi) w_2 \nabla c + D(\psi) \nabla w_3 \right) \\
& \quad + \operatorname{div} \left(\left(\frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} w_2 + \frac{\partial \mathcal{A}_3(\psi, c)}{\partial c} w_3 \right) \nabla \psi + \mathcal{A}_3(\psi, c) \nabla w_2 \right), \quad \text{on } \mathcal{Q} \\
& (\mathbf{w}_1, w_2, w_3)(t=0) = (\mathbf{0}, 0, 0), \quad \text{in } \Omega \\
& \mathbf{w}_1 = \mathbf{0}, \quad \frac{\partial w_2}{\partial \mathbf{n}} = 0, \quad \frac{\partial w_3}{\partial \mathbf{n}} = 0. \quad \text{on } \Sigma.
\end{aligned} \tag{5.8}$$

Proof: The problem (5.8) is similar to the problem (5.1). By using a similar arguments as in the proof of Theorem 1 and the regularity of \mathbf{X} , we can obtain the existence and uniqueness of the solution $\mathbf{Y} \in \mathcal{W}$ of (5.8).

For more detail about the proof of the differentiability results of \mathcal{F} , the reader is referred to [6] and Chapter 11 in [8]. \square

We present here a lemma which we need to give the optimality conditions.

Lemma 5 *Let \mathbf{f} , \mathbf{g} , \mathbf{h} and $\mathbf{k} \in \mathbb{R}^2$. Then we have the following relation*

$$((\mathbf{f} \times \mathbf{g}) \times \mathbf{h}) \cdot \mathbf{k} = (\mathbf{g} \times (\mathbf{h} \times \mathbf{k})) \cdot \mathbf{f}.$$

Proof: The vector triple product defined in the above expression is given explicitly by (2.1). To prove the above result we consider the left-hand-side of the expression

$$\begin{aligned}
& ((\mathbf{f} \times \mathbf{g}) \times \mathbf{h}) \cdot \mathbf{k} = \mathbf{k} \cdot ((\mathbf{f} \times \mathbf{g}) \times \mathbf{h}) \\
& = (\mathbf{f} \times \mathbf{g}) \cdot (\mathbf{h} \times \mathbf{k}) \\
& = (\mathbf{h} \times \mathbf{k}) \cdot (\mathbf{f} \times \mathbf{g}) \\
& = (\mathbf{g} \times (\mathbf{h} \times \mathbf{k})) \cdot \mathbf{f}.
\end{aligned}$$

which is the required expression. \square

Theorem 7 *Under the hypothesis of Theorem 6. Let $\mathbf{B}^* \in \mathcal{U}_a$ be an optimal control defined by (5.6) and $\mathbf{X}^* \in \mathcal{W}$ be the optimal state such that $\mathbf{X}^* = \mathcal{F}(\mathbf{B}^*)$ is the solution of (5.1). Then there exists a unique solution $\tilde{\mathbf{X}} = (\tilde{\mathbf{u}}, \tilde{\psi}, \tilde{c}) \in \mathcal{W}$ for the following adjoint problem corresponding to the primal problem (5.1)*

$$\begin{aligned}
& -\frac{\partial \tilde{\mathbf{u}}}{\partial t} - (\mathbf{u}^* \cdot \nabla) \tilde{\mathbf{u}} + (\nabla \mathbf{u}^*)^{tr} \tilde{\mathbf{u}} + \tilde{\psi} \nabla \psi^* + \tilde{c} \nabla c^* = -\nabla \tilde{p} + \alpha \Delta \tilde{\mathbf{u}} \\
& \quad + b(\psi^*)(\mathbf{B}^* \times (\mathbf{B}^* \times \tilde{\mathbf{u}})) + \alpha_1(\mathbf{u}^* - \mathbf{u}_{obs}) \quad \text{on } \mathcal{Q} \\
& \operatorname{div}(\tilde{\mathbf{u}}) = 0. \quad \text{on } \mathcal{Q} \\
& -\frac{\partial \tilde{\psi}}{\partial t} - (\mathbf{u}^* \cdot \nabla) \tilde{\psi} = \epsilon_2 \Delta \tilde{\psi} - \frac{\partial \mathcal{A}_2(\psi^*, c^*)}{\partial \psi^*} \tilde{\psi} + b'(\psi^*)((\mathbf{u}^* \times \mathbf{B}^*) \times \mathbf{B}^*) \cdot \tilde{\mathbf{u}} \\
& \quad + \frac{\partial \mathcal{A}_1(\psi^*, c^*)}{\partial \psi^*} \cdot \tilde{\mathbf{u}} - D'(\psi^*) \nabla c^* \cdot \nabla \tilde{c} - \frac{\partial \mathcal{A}_3(\psi^*, c^*)}{\partial \psi^*} \nabla \psi^* \cdot \nabla \tilde{c} \\
& \quad + \operatorname{div}(\mathcal{A}_3(\psi^*, c^*) \nabla \tilde{c}) + \kappa a'_1(\psi^*) \mathbf{B}^* \tilde{\mathbf{u}} + \alpha_2(\psi^* - \psi_{obs}) \quad \text{on } \mathcal{Q} \\
& -\frac{\partial \tilde{c}}{\partial t} - (\mathbf{u}^* \cdot \nabla) \tilde{c} = \operatorname{div}(D(\psi^*) \nabla \tilde{c}) - \frac{\partial \mathcal{A}_3(\psi^*, c^*)}{\partial c^*} \nabla \psi^* \cdot \nabla \tilde{c} - \frac{\partial \mathcal{A}_2(\psi^*, c^*)}{\partial c^*} \tilde{\psi} \\
& \quad + \frac{\partial \mathcal{A}_1(\psi^*, c^*)}{\partial c^*} \cdot \tilde{\mathbf{u}} + \alpha_3(c^* - c_{obs}) \quad \text{on } \mathcal{Q}
\end{aligned} \tag{5.9}$$

with the final conditions

$$(\tilde{\mathbf{u}}, \tilde{\psi}, \tilde{c})(t = T_f) = (\mathbf{0}, 0, 0), \quad \text{in } \Omega$$

and the boundary conditions

$$\tilde{\mathbf{u}} = \mathbf{0}, \quad \frac{\partial \tilde{\psi}}{\partial \mathbf{n}} = 0, \quad \frac{\partial \tilde{c}}{\partial \mathbf{n}} = 0. \quad \text{on } \Sigma$$

Furthermore, we have $(\forall \mathbf{B} \in \mathcal{U}_a)$

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \left(b(\psi^*)((\mathbf{B}^* \times \tilde{\mathbf{u}}) \times \mathbf{u}^* + \tilde{\mathbf{u}} \times (\mathbf{u}^* \times \mathbf{B}^*)) + \kappa a_1(\psi^*) \tilde{\mathbf{u}} \right. \\
& \quad \left. + \beta \mathbf{B}^* \right) \cdot (\mathbf{B} - \mathbf{B}^*) \, d\mathbf{x} dt \geq 0.
\end{aligned} \tag{5.10}$$

Proof: The problem (5.9) admits a unique solution $\tilde{\mathbf{X}} = (\tilde{\mathbf{u}}, \tilde{\psi}, \tilde{c})$ in \mathcal{W} , since the observation $\mathbf{X}_{obs} = (\mathbf{u}_{obs}, \psi_{obs}, c_{obs}) \in L^2(\mathcal{Q})$. To prove this result, we change the variables of this problem by reversing the time variable, *i.e.*, $t = T_f - t$ where T_f is the final time, and we apply the same technique to obtain the existence and uniqueness as in Theorems 1 and 4.

The cost functional J defined by (5.6) is composed of G-differentiable functions, consequently J is G-differentiable and its G-derivative can be given by differentiating (5.6) with respect to \mathbf{B} in the direction \mathcal{B} as

$$\begin{aligned}
J'(\mathbf{B}) \cdot \mathcal{B} &= \alpha_1 \int_0^{T_f} \int_{\Omega} (\mathbf{u} - \mathbf{u}_{obs}) \cdot \mathbf{w}_1 \, d\mathbf{x} dt + \alpha_2 \int_0^{T_f} \int_{\Omega} (\psi - \psi_{obs}) w_2 \, d\mathbf{x} dt \\
& \quad + \alpha_3 \int_0^{T_f} \int_{\Omega} (c - c_{obs}) w_3 \, d\mathbf{x} dt + \beta \int_0^{T_f} \int_{\Omega} \mathbf{B} \cdot \mathcal{B} \, d\mathbf{x} dt
\end{aligned} \tag{5.11}$$

where $\mathbf{w} = (\mathbf{w}_1, w_2, w_3) = \mathcal{F}'(\mathbf{B}) \cdot \mathbf{B}$ is the solution of (5.8).

Now multiplying the first equation of (5.8) by a sufficiently regular function $\tilde{\mathbf{u}}$, such that $\tilde{\mathbf{u}}(T_f) = 0$ on both sides and then integrating over \mathcal{Q} , we have

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \frac{\partial \mathbf{w}_1}{\partial t} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{w}_1 \cdot \nabla) \mathbf{u} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{w}_1 \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt \\
&= - \int_0^{T_f} \int_{\Omega} \nabla \pi \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt + \alpha \int_0^{T_f} \int_{\Omega} \Delta \mathbf{w}_1 \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} \left\{ \frac{\partial \mathcal{A}_1(\psi, c)}{\partial \psi} \cdot w_2 + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial c} \cdot w_3 \right\} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} b'(\psi) w_2 ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} b(\psi) \left\{ (\mathbf{w}_1 \times \mathbf{B}) \times \mathbf{B} \right. \\
&+ (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} + (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \left. \right\} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} \left(\kappa a'_1(\psi) \mathbf{B} w_2 + \kappa a_1(\psi) \mathbf{B} \right) \tilde{\mathbf{u}} \, d\mathbf{x} dt.
\end{aligned}$$

Using Green's theorem and Lemma 5, the above equation takes the following form

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \frac{\partial \mathbf{w}_1}{\partial t} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{w}_1 \cdot \nabla) \mathbf{u} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{w}_1 \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt \\
&= \int_0^{T_f} \int_{\Omega} \pi \operatorname{div}(\tilde{\mathbf{u}}) \, d\mathbf{x} dt - \int_0^{T_f} \int_{\Gamma} \pi \tilde{\mathbf{u}} \cdot \mathbf{n} \, d\Gamma dt + \alpha \int_0^{T_f} \int_{\Omega} \Delta \tilde{\mathbf{u}} \cdot \mathbf{w}_1 \, d\mathbf{x} dt \\
&- \alpha \int_0^{T_f} \int_{\Gamma} \mathbf{w}_1 \cdot \nabla \tilde{\mathbf{u}} \cdot \mathbf{n} \, d\Gamma dt + \int_0^{T_f} \int_{\Omega} \left\{ \frac{\partial \mathcal{A}_1(\psi, c)}{\partial \psi} \cdot w_2 + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial c} \cdot w_3 \right\} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} \left(b'(\psi) ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \cdot \tilde{\mathbf{u}} + \kappa a'_1(\psi) \mathbf{B} \tilde{\mathbf{u}} \right) w_2 \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} b(\psi) (\mathbf{B} \times (\mathbf{B} \times \tilde{\mathbf{u}})) \cdot \mathbf{w}_1 \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} \left(b(\psi) \left\{ (\mathbf{B} \times \tilde{\mathbf{u}}) \times \mathbf{u} + \tilde{\mathbf{u}} \times (\mathbf{u} \times \mathbf{B}) \right\} + \kappa a_1(\psi) \tilde{\mathbf{u}} \right) \cdot \mathbf{B} \, d\mathbf{x} dt
\end{aligned}$$

Integrating the first term of the above equation with respect to t and as $\tilde{\mathbf{u}}(T_f) = 0$, $\mathbf{w}_1(0) = 0$ and using Lemma 2, the above equation yields

$$\begin{aligned}
& - \int_0^{T_f} \int_{\Omega} \frac{\partial \tilde{\mathbf{u}}}{\partial t} \cdot \mathbf{w}_1 \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\nabla \mathbf{u})^{tr} \tilde{\mathbf{u}} \cdot \mathbf{w}_1 \, d\mathbf{x} dt - \int_0^{T_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \tilde{\mathbf{u}} \cdot \mathbf{w}_1 \, d\mathbf{x} dt \\
& = \int_0^{T_f} \int_{\Omega} \pi \operatorname{div}(\tilde{\mathbf{u}}) \, d\mathbf{x} dt - \int_0^{T_f} \int_{\Gamma} \pi \tilde{\mathbf{u}} \cdot \mathbf{n} \, d\Gamma dt + \alpha \int_0^{T_f} \int_{\Omega} \Delta \tilde{\mathbf{u}} \cdot \mathbf{w}_1 \, d\mathbf{x} dt \\
& - \alpha \int_0^{T_f} \int_{\Gamma} \mathbf{w}_1 \cdot \nabla \tilde{\mathbf{u}} \cdot \mathbf{n} \, d\Gamma dt + \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_1(\psi, c)}{\partial \psi} \cdot \tilde{\mathbf{u}} \, w_2 \, d\mathbf{x} dt \\
& + \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_1(\psi, c)}{\partial c} \cdot \tilde{\mathbf{u}} \, w_3 \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} b(\psi) (\mathbf{B} \times (\mathbf{B} \times \tilde{\mathbf{u}})) \cdot \mathbf{w}_1 \, d\mathbf{x} dt \\
& + \int_0^{T_f} \int_{\Omega} \left(b'(\psi) ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \cdot \tilde{\mathbf{u}} + \kappa a'_1(\psi) \mathbf{B} \tilde{\mathbf{u}} \right) w_2 \, d\mathbf{x} dt \\
& + \int_0^{T_f} \int_{\Omega} b(\psi) \left\{ (\mathbf{B} \times \tilde{\mathbf{u}}) \times \mathbf{u} + \tilde{\mathbf{u}} \times (\mathbf{u} \times \mathbf{B}) \right\} \cdot \mathbf{B} \, d\mathbf{x} dt.
\end{aligned} \tag{5.12}$$

Multiplying now the third equation of (5.8) by a sufficiently regular function $\tilde{\psi}$, such that $\tilde{\psi}(T_f) = 0$, on both sides and then integrating over \mathcal{Q} , we have

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \frac{\partial w_2}{\partial t} \tilde{\psi} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{w}_1 \cdot \nabla) \psi \tilde{\psi} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) w_2 \tilde{\psi} \, d\mathbf{x} dt \\
& = \epsilon_2 \int_0^{T_f} \int_{\Omega} \Delta w_2 \tilde{\psi} \, d\mathbf{x} dt - \int_0^{T_f} \int_{\Omega} \left\{ \frac{\partial \mathcal{A}_2(\psi, c)}{\partial \psi} w_2 + \frac{\partial \mathcal{A}_2(\psi, c)}{\partial c} w_3 \right\} \tilde{\psi} \, d\mathbf{x} dt
\end{aligned}$$

Integrating the first term of the above equation with respect to t and using $\tilde{\psi}(T_f) = 0$, $w_2(0) = 0$ and Green's formula, the above equation can finally be written as

$$\begin{aligned}
& - \int_0^{T_f} \int_{\Omega} \frac{\partial \tilde{\psi}}{\partial t} w_2 \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} \tilde{\psi} \nabla \psi \cdot \mathbf{w}_1 \, d\mathbf{x} dt - \int_0^{T_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \tilde{\psi} w_2 \, d\mathbf{x} dt \\
& = \epsilon_2 \int_0^{T_f} \int_{\Omega} \Delta \tilde{\psi} w_2 \, d\mathbf{x} dt - \epsilon_2 \int_0^{T_f} \int_{\Gamma} w_2 \nabla \tilde{\psi} \cdot \mathbf{n} \, d\Gamma dt \\
& - \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_2(\psi, c)}{\partial \psi} \tilde{\psi} w_2 \, d\mathbf{x} dt - \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_2(\psi, c)}{\partial c} \tilde{\psi} w_3 \, d\mathbf{x} dt.
\end{aligned} \tag{5.13}$$

Multiplying now fourth equation of the problem (5.8) by a sufficiently regular function \tilde{c} , such that $\tilde{c}(T_f) = 0$, on both sides and then integrating over \mathcal{Q} , we have

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \frac{\partial w_3}{\partial t} \cdot \tilde{c} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{w}_1 \cdot \nabla) c \cdot \tilde{c} \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) w_3 \cdot \tilde{c} \, d\mathbf{x} dt \\
&= \int_0^{T_f} \int_{\Omega} \operatorname{div} \left(D'(\psi) w_2 \nabla c + D(\psi) \nabla w_3 \right) \cdot \tilde{c} \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} \operatorname{div} \left(\left(\frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} w_2 + \frac{\partial \mathcal{A}_3(\psi, c)}{\partial c} w_3 \right) \nabla \psi + \mathcal{A}_3(\psi, c) \nabla w_2 \right) \cdot \tilde{c} \, d\mathbf{x} dt.
\end{aligned} \tag{5.14}$$

Consider the following integral

$$\begin{aligned}
\int_0^{T_f} \int_{\Omega} \operatorname{div} (D'(\psi) \nabla c \tilde{c} w_2) \, d\mathbf{x} dt &= \int_0^{T_f} \int_{\Omega} \operatorname{div} (D'(\psi) \nabla c w_2) \tilde{c} \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} D'(\psi) \nabla c \cdot \nabla \tilde{c} w_2 \, d\mathbf{x} dt.
\end{aligned}$$

By employing the divergence theorem and as $\nabla c \cdot \mathbf{n} = 0$, the above equation becomes

$$\int_0^{T_f} \int_{\Omega} \operatorname{div} (D'(\psi) \nabla c w_2) \tilde{c} \, d\mathbf{x} dt = - \int_0^{T_f} \int_{\Omega} D'(\psi) \nabla c \cdot \nabla \tilde{c} w_2 \, d\mathbf{x} dt. \tag{5.15}$$

Consider now the following integral

$$\begin{aligned}
\int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla w_3 \tilde{c}) \, d\mathbf{x} dt &= \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla w_3) \tilde{c} \, d\mathbf{x} dt \\
&+ \int_0^{T_f} \int_{\Omega} D(\psi) \nabla w_3 \cdot \nabla \tilde{c} \, d\mathbf{x} dt.
\end{aligned}$$

Using again the divergence theorem and using $\nabla w_3 \cdot \mathbf{n} = 0$, we arrive at

$$\int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla w_3) \tilde{c} \, d\mathbf{x} dt = - \int_0^{T_f} \int_{\Omega} D(\psi) \nabla w_3 \cdot \nabla \tilde{c} \, d\mathbf{x} dt. \tag{5.16}$$

Again consider the following integral

$$\begin{aligned} \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla \tilde{c} w_3) \, d\mathbf{x} dt &= \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla \tilde{c}) w_3 \, d\mathbf{x} dt \\ &\quad + \int_0^{T_f} \int_{\Omega} D(\psi) \nabla w_3 \cdot \nabla \tilde{c} \, d\mathbf{x} dt. \end{aligned}$$

Applying the divergence theorem on the left-hand-side of the above equation, we arrive at

$$\begin{aligned} \int_0^{T_f} \int_{\Gamma} D(\psi) w_3 \nabla \tilde{c} \cdot \mathbf{n} \, d\Gamma dt &= \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla \tilde{c}) w_3 \, d\mathbf{x} dt \\ &\quad + \int_0^{T_f} \int_{\Omega} D(\psi) \nabla w_3 \cdot \nabla \tilde{c} \, d\mathbf{x} dt, \end{aligned}$$

and then

$$\begin{aligned} \int_0^{T_f} \int_{\Omega} D(\psi) \nabla w_3 \cdot \nabla \tilde{c} \, d\mathbf{x} dt &= - \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla \tilde{c}) w_3 \, d\mathbf{x} dt \\ &\quad + \int_0^{T_f} \int_{\Gamma} D(\psi) w_3 \nabla \tilde{c} \cdot \mathbf{n} \, d\Gamma dt. \end{aligned} \quad (5.17)$$

Making use of equation (5.17) in equation (5.16), we obtain

$$\begin{aligned} \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla w_3) \tilde{c} \, d\mathbf{x} dt &= \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla \tilde{c}) w_3 \, d\mathbf{x} dt \\ &\quad - \int_0^{T_f} \int_{\Gamma} D(\psi) w_3 \nabla \tilde{c} \cdot \mathbf{n} \, d\Gamma dt. \end{aligned} \quad (5.18)$$

Similarly, we can derive the following equations

$$\begin{aligned} \int_0^{T_f} \int_{\Omega} \operatorname{div} (\mathcal{A}_3(\psi, c) \nabla w_2) \tilde{c} \, d\mathbf{x} dt &= \int_0^{T_f} \int_{\Omega} \operatorname{div} (\mathcal{A}_3(\psi, c) \nabla \tilde{c}) w_2 \, d\mathbf{x} dt \\ &\quad - \int_0^{T_f} \int_{\Gamma} \mathcal{A}_3(\psi, c) w_2 \nabla \tilde{c} \cdot \mathbf{n} \, d\Gamma dt, \end{aligned} \quad (5.19)$$

$$\begin{aligned}
\int_0^{T_f} \int_{\Omega} \operatorname{div} \left(\frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \psi \, w_2 \right) \tilde{c} \, d\mathbf{x} dt &= - \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \tilde{c} \cdot \nabla \psi \, w_2 \, d\mathbf{x} dt \\
&\quad + \int_0^{T_f} \int_{\Gamma} \frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \psi \, \tilde{c} \, w_2 \cdot \mathbf{n} \, d\Gamma dt, \quad (5.20)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{T_f} \int_{\Omega} \operatorname{div} \left(\frac{\partial \mathcal{A}_3(\psi, c)}{\partial c} \nabla \psi \, w_3 \right) \tilde{c} \, d\mathbf{x} dt &= - \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \tilde{c} \cdot \nabla \psi \, w_3 \, d\mathbf{x} dt \\
&\quad + \int_0^{T_f} \int_{\Gamma} \frac{\partial \mathcal{A}_3(\psi, c)}{\partial c} \nabla \psi \, \tilde{c} \, w_3 \cdot \mathbf{n} \, d\Gamma dt. \quad (5.21)
\end{aligned}$$

Using equations (5.15) and (5.18)-(5.21) in the equation (5.14) and then integrating with respect to time t . We obtain, by using $\tilde{c}(T_f) = 0$ and $w_3(0) = 0$,

$$\begin{aligned}
& - \int_0^{T_f} \int_{\Omega} \frac{\partial \tilde{c}}{\partial t} w_3 \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} \tilde{c} \nabla c \cdot \mathbf{w}_1 \, d\mathbf{x} dt - \int_0^{T_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \tilde{c} w_3 \, d\mathbf{x} dt \\
&= - \int_0^{T_f} \int_{\Omega} D'(\psi) \nabla c \cdot \nabla \tilde{c} w_2 \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} \operatorname{div} (D(\psi) \nabla \tilde{c}) w_3 \, d\mathbf{x} dt \\
&\quad - \int_0^{T_f} \int_{\Gamma} D(\psi) w_3 \, \nabla \tilde{c} \cdot \mathbf{n} \, d\Gamma dt - \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \psi \cdot \nabla \tilde{c} w_2 \, d\mathbf{x} dt \quad (5.22) \\
&\quad - \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_3(\psi, c)}{\partial c} \nabla \psi \cdot \nabla \tilde{c} w_3 \, d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} \operatorname{div} (\mathcal{A}_3(\psi, c) \nabla \tilde{c}) w_2 \, d\mathbf{x} dt \\
&\quad - \int_0^{T_f} \int_{\Gamma} \mathcal{A}_3(\psi, c) w_2 \, \nabla \tilde{c} \cdot \mathbf{n} \, d\Gamma dt
\end{aligned}$$

In order to simplify equations (5.12), (5.13) and (5.22), we suppose that $(\tilde{\mathbf{u}}, \tilde{\psi}, \tilde{c})$ satisfy the following system

$$\begin{aligned}
& -\frac{\partial \tilde{\mathbf{u}}}{\partial t} - (\mathbf{u} \cdot \nabla) \tilde{\mathbf{u}} + (\nabla \mathbf{u})^{tr} \tilde{\mathbf{u}} + \tilde{\psi} \nabla \psi + \tilde{c} \nabla c = -\nabla \tilde{p} + \alpha \Delta \tilde{\mathbf{u}} \\
& \quad + b(\psi)(\mathbf{B} \times (\mathbf{B} \times \tilde{\mathbf{u}})) + \alpha_1(\mathbf{u} - \mathbf{u}_{obs}), \quad \text{on } \mathcal{Q} \\
& \text{div}(\tilde{\mathbf{u}}) = 0, \quad \text{on } \mathcal{Q} \\
& -\frac{\partial \tilde{\psi}}{\partial t} - (\mathbf{u} \cdot \nabla) \tilde{\psi} = \epsilon_2 \Delta \tilde{\psi} - \frac{\partial \mathcal{A}_2(\psi, c)}{\partial \psi} \tilde{\psi} + b'(\psi)((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \cdot \tilde{\mathbf{u}} \\
& \quad + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial \psi} \cdot \tilde{\mathbf{u}} - D'(\psi) \nabla c \cdot \nabla \tilde{c} - \frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \psi \cdot \nabla \tilde{c} \\
& \quad + \text{div}(\mathcal{A}_3(\psi, c) \nabla \tilde{c}) + \kappa a'_1(\psi) \mathbf{B} \tilde{\mathbf{u}} + \alpha_2(\psi - \psi_{obs}), \quad \text{on } \mathcal{Q} \\
& -\frac{\partial \tilde{c}}{\partial t} - (\mathbf{u} \cdot \nabla) \tilde{c} = \text{div}(D(\psi) \nabla \tilde{c}) - \frac{\partial \mathcal{A}_3(\psi, c)}{\partial c} \nabla \psi \cdot \nabla \tilde{c} - \frac{\partial \mathcal{A}_2(\psi, c)}{\partial c} \tilde{\psi} \\
& \quad + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial c} \cdot \tilde{\mathbf{u}} + \alpha_3(c - c_{obs}), \quad \text{on } \mathcal{Q} \\
& (\tilde{\mathbf{u}}, \tilde{\psi}, \tilde{c})(t = T_f) = (\mathbf{0}, 0, 0), \quad \text{in } \Omega \\
& \tilde{\mathbf{u}} = \mathbf{0}, \quad \frac{\partial \tilde{\psi}}{\partial \mathbf{n}} = 0, \quad \frac{\partial \tilde{c}}{\partial \mathbf{n}} = 0. \quad \text{on } \Sigma
\end{aligned} \tag{5.23}$$

Making use of (5.23), the equations (5.12), (5.13) and (5.22) becomes

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \left(\alpha_1(\mathbf{u} - \mathbf{u}_{obs}) - \tilde{\psi} \nabla \psi - \tilde{c} \nabla c \right) \cdot \mathbf{w}_1 d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} \left(-\frac{\partial \mathcal{A}_1(\psi, c)}{\partial \psi} \cdot \tilde{\mathbf{u}} - \kappa a'_1(\psi) \mathbf{B} \tilde{\mathbf{u}} \right. \\
& \quad \left. - b'(\psi)((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \cdot \tilde{\mathbf{u}} \right) w_2 d\mathbf{x} dt - \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_1(\psi, c)}{\partial c} \cdot \tilde{\mathbf{u}} w_3 d\mathbf{x} dt \tag{5.24} \\
& = \int_0^{T_f} \int_{\Omega} \left(b(\psi) \left\{ (\mathbf{B} \times \tilde{\mathbf{u}}) \times \mathbf{u} + \tilde{\mathbf{u}} \times (\mathbf{u} \times \mathbf{B}) \right\} + \kappa a_1(\psi) \tilde{\mathbf{u}} \right) \cdot \mathcal{B} d\mathbf{x} dt,
\end{aligned}$$

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \left(\alpha_2(\psi - \psi_{obs}) + b'(\psi)((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \cdot \tilde{\mathbf{u}} + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial \psi} \cdot \tilde{\mathbf{u}} - \frac{\partial D}{\partial \psi} \nabla c \cdot \nabla \tilde{c} \right. \\
& \quad \left. - \frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \psi \cdot \nabla \tilde{c} + \text{div}(\mathcal{A}_3(\psi, c) \nabla \tilde{c}) + \kappa a'_1(\psi) \mathbf{B} \tilde{\mathbf{u}} \right) w_2 d\mathbf{x} dt \tag{5.25} \\
& + \int_0^{T_f} \int_{\Omega} \tilde{\psi} \nabla \psi \cdot \mathbf{w}_1 d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} \frac{\partial \mathcal{A}_2(\psi, c)}{\partial c} \tilde{\psi} w_3 d\mathbf{x} dt = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \left(\alpha_3 (c - c_{obs}) - \frac{\partial \mathcal{A}_2(\psi, c)}{\partial c} \tilde{\psi} + \frac{\partial \mathcal{A}_1(\psi, c)}{\partial c} \cdot \tilde{\mathbf{u}} \right) w_3 d\mathbf{x} dt \\
& + \int_0^{T_f} \int_{\Omega} \tilde{c} \nabla c \cdot \mathbf{w}_1 d\mathbf{x} dt + \int_0^{T_f} \int_{\Omega} \left(D'(\psi) \nabla c \cdot \nabla \tilde{c} + \frac{\partial \mathcal{A}_3(\psi, c)}{\partial \psi} \nabla \psi \cdot \nabla \tilde{c} \right. \\
& \left. - \operatorname{div} (\mathcal{A}_3(\psi, c) \nabla \tilde{c}) \right) w_2 d\mathbf{x} dt = 0.
\end{aligned} \tag{5.26}$$

Adding the respective sides of the equations (5.24), (5.25) and (5.26), we finally obtain

$$\begin{aligned}
& \alpha_1 \int_0^{T_f} \int_{\Omega} (\mathbf{u} - \mathbf{u}_{obs}) \cdot \mathbf{w}_1 d\mathbf{x} dt + \alpha_2 \int_0^{T_f} \int_{\Omega} (\psi - \psi_{obs}) w_2 d\mathbf{x} dt \\
& + \alpha_3 \int_0^{T_f} \int_{\Omega} (c - c_{obs}) w_3 d\mathbf{x} dt = \int_0^{T_f} \int_{\Omega} \left(b(\psi) \left\{ (\mathbf{B} \times \tilde{\mathbf{u}}) \times \mathbf{u} \right. \right. \\
& \left. \left. + \tilde{\mathbf{u}} \times (\mathbf{u} \times \mathbf{B}) \right\} + \kappa a_1(\psi) \tilde{\mathbf{u}} \right) \cdot \mathbf{B} d\mathbf{x} dt.
\end{aligned} \tag{5.27}$$

According to (5.27), the equation (5.11) takes the form

$$J'(\mathbf{B}) \cdot \mathbf{B} = \int_0^{T_f} \int_{\Omega} \left(b(\psi) \left\{ (\mathbf{B} \times \tilde{\mathbf{u}}) \times \mathbf{u} + \tilde{\mathbf{u}} \times (\mathbf{u} \times \mathbf{B}) \right\} + \kappa a_1(\psi) \tilde{\mathbf{u}} + \beta \mathbf{B} \right) \cdot \mathbf{B} d\mathbf{x} dt.$$

Since \mathbf{B}^* is an optimal solution of J , we have

$$J'(\mathbf{B}^*) \cdot (\mathbf{B}^* - \mathbf{B}) \geq 0,$$

and then

$$\begin{aligned}
& \int_0^{T_f} \int_{\Omega} \left(b(\psi^*) \left((\mathbf{B}^* \times \tilde{\mathbf{u}}^*) \times \mathbf{u}^* + \tilde{\mathbf{u}}^* \times (\mathbf{u}^* \times \mathbf{B}^*) \right) + \kappa a_1(\psi) \tilde{\mathbf{u}} \right. \\
& \left. + \beta \mathbf{B}^* \right) \cdot (\mathbf{B} - \mathbf{B}^*) d\mathbf{x} dt \geq 0.
\end{aligned}$$

where $\tilde{\mathbf{X}}^*$ is the solution of (5.23) corresponding to the primal solution $\mathbf{X}^* = \mathcal{F}(\mathbf{B}^*)$. This completes the proof. \square

5.4 Remarks on the numerical implementation

In this section, we shall give remarks on the numerical resolution of the optimization problem (5.7) by sequentially solving the primal problem (5.1) and the corresponding adjoint problem (5.9) and updating the control by a Gradient based iterative

algorithm. The individual primal and adjoint systems can be solved using same discretization and technique as given in Chapter 3. The main difficulty in the application of this algorithm is to realize the reversed time directions of the primal problem in the corresponding adjoint problem. Gradient algorithm for the resolution of the optimization problem (5.7) can be described as follows.

We denote by k , the iteration index and \mathbf{B}_k the numerical approximation of the magnetic-field (control variable) at the k th iteration of the algorithm. The different steps involved in this algorithm are described below

- I Initialization: $k = 1$ and \mathbf{B}_0 .
- II Resolution of the direct problem (5.1) with the source term \mathbf{B}_k , gives $\mathbf{X}_k = \mathcal{F}(\mathbf{B}_k)$.
- III Resolution of the adjoint problem (5.9), by giving \mathbf{X}_k , gives $\tilde{\mathbf{X}}_k$.
- IV Calculation of the Gradient expression of J at point \mathbf{B}_k

$$J'(\mathbf{B}_k) = b(\psi_k) \left\{ (\mathbf{B}_k \times \tilde{\mathbf{u}}_k) \times \mathbf{u}_k + \tilde{\mathbf{u}}_k \times (\mathbf{u}_k \times \mathbf{B}_k) \right\} + \kappa a_1(\psi_k) \tilde{\mathbf{u}}_k + \beta \mathbf{B}_k.$$
- V Calculation of \mathbf{B}_{k+1}

$$\mathbf{B}_{k+1} = \mathbf{B}_k - \lambda_k J'(\mathbf{B}_k).$$
- VI If the gradient is sufficiently small (we have the convergence) then stop. Else by setting $k = k + 1$, repeat from the second step until the required convergence is achieved. The approximation of the optimal solution $(\mathbf{B}^*, \mathbf{X}^*)$ is then given by $(\mathbf{B}_k, \mathbf{X}_k)$.

5.5 Conclusion

We have successfully formulated the optimal control of the problem (5.1) and established the existence results and optimality conditions along with the adjoint problem. Next we have introduced a Gradient algorithm for the numerical resolution of the optimization problem (5.7). We are trying to implement this algorithm in Comsol together with MatLab. We have already succeeded to solve the coupled direct and the corresponding adjoint problems. The major difficulty in the implementation of this technique is that we have to substitute reversed time solution of the direct problem for the resolution of the adjoint problem.

Conclusion

In this work, we have developed a new phase-field model that incorporate convection together with the influence of magnetic-field. For the theoretical and numerical study, we have considered the case of 2D isothermal solidification model. Then we have established the existence, regularity, stability and uniqueness results for the isotropic model. For the numerical simulations, we have worked with the isotropic case and the general anisotropic model. We have developed a numerical scheme and demonstrated the convergence and stability of the scheme for both isotropic and anisotropic models with the help of various examples. The realistic numerical simulations has been carried out by choosing the real physical parameters of the binary mixture Ni-Cu in order to fit a realistic physical alloy. We have focused mainly the effect of magnetic-field on the growth of dendrites during the solidification process by considering various magnetic-fields (all other parameters remain fixed). We have found that the constant magnetic-field does not effect considerably but the variable magnetic-field has a significant effect on the structure of dendrites and on the dynamics of the melt flow. These observations are in good agreement with the study made by [48].

Further we have formulated the optimal control for the isotropic model using the control in magnetic field. We have established the existence results of the optimal solution and presented the optimality conditions together with adjoint problem corresponding to the primal problem. Currently, we are working to implement the Gradient algorithm, given in the chapter 5, in Comsol together with the MatLab.

Several questions remain open for the future investigations. The realistic physical simulations can be reviewed for the adaptive meshes. It also remains to investigate the effect of different physical parameters, like interface thickness, anisotropy amplitude, on the dendrite growth during the solidification process. Further, we can introduce a stochastic noise at the interface to stimulate fluctuations at the interface in the model which give rise to the realistic structure of dendrites observed in practical situations. The simulations can be broaden for the non-isothermal anisotropic case by the inclusion of temperature equation. For the better understanding of magnetic-field

influence on the microstructure of dendrites, the simulations can be extended to the three dimensions.

Appendices

Appendix A: Description of DASSL

In this appendix, we shall describe the method used by time-dependent solver in COMSOL Multiphysics 3.4 to solve the model problem. Time dependent solver in COMSOL uses the program DASPK (and DASSL) written in computer language Fortran by P. N. Brown et. al. [50], where DASPK is an extension of the program DASSL written in Fortran by Linda R. Petzold [37]. We have used DASSL for the resolution of our model, therefore we shall describe the program DASSL.

The program DASSL is developed for the numerical solution of the implicit systems of differential/algebraic equations written in the form

$$F(t, y, y') = 0, \quad y(t_0) = y_0. \quad (.1)$$

where F , y and y' are N dimensional vectors and y' represents the derivative with respect to time. DASSL can solve two types of problems. The first type problems for which it is not possible to solve for y' explicitly to rewrite equation (.1) in the form of a standard ODE system $y' = f(t, y)$. The second class of problems for which it is possible in theory to solve for y' , but is impractical to do so. The technique used in DASSL to solve differential/algebraic equations is based on the idea of Gear [12] in which the derivative in equation (.1) is replaced by the back-ward difference Euler formula to obtain

$$F(t_n, y_n, \frac{y_n - y_{n-1}}{\Delta t_n}) = 0. \quad (.2)$$

The resulting equation is then solved using Newton's method as

$$y_n^{m+1} = y_n^m - G^{-1} F(t_n, y_n^m, \frac{y_n^m - y_{n-1}}{\Delta t_n}) = 0. \quad (.3)$$

where

$$G = \left(\frac{\partial F}{\partial y'} + \frac{1}{\Delta t_n} \frac{\partial F}{\partial y} \right)$$

and m is the iteration index.

The techniques used in DASSL are the extension of this method. DASSL program

uses a k th order variable back-ward Euler formula where k varies between 1 to 5. Here variable order back-ward formula means that on each step DASSL chooses an appropriate order k and the step size Δt_n depending upon the behavior of the solution. Further details about the choice of step size and order will be given later. To explain the algorithm used in DASSL, it is convenient to write the equation (.2) in simplified form as

$$F(t, y, \hat{\alpha}_1 y + \hat{\beta}) = 0. \quad (.4)$$

where $\hat{\alpha}_1$ is a constant which changes whenever the step size or order changes, $\hat{\beta}$ is a vector which depends on the solution at the previous times and $t, y, \hat{\alpha}_1, \hat{\beta}$ are evaluated at t_n . The code DASSL solves the equation (.4) by using the modified version of Newton method given by

$$y^{m+1} = y^m - \hat{\gamma} G^{-1} F(t, y^m, \hat{\alpha}_1 y^m + \hat{\beta}) = 0. \quad (.5)$$

where the iteration matrix G takes now the form

$$G = \left(\frac{\partial F}{\partial y'} + \hat{\alpha}_2 \frac{1}{\Delta t_n} \frac{\partial F}{\partial y} \right)$$

which is computed and factored and then it is used for as many steps as possible. In the iteration matrix the value of the constant $\hat{\alpha}_2$ when G was last calculated is generally different from the current value of the constant $\hat{\alpha}_1$ in equation (.5). If the values of these constants differ significantly then the convergence of (.5) is not guaranteed. And the constant $\hat{\gamma}$ in (.5) is chosen to speed up the convergence when $\hat{\alpha}_1 \neq \hat{\alpha}_2$ as

$$\hat{\gamma} = \frac{2}{1 + \hat{\alpha}_1 / \hat{\alpha}_2} \quad (.6)$$

The convergence rate of equation (.5) is estimated by

$$\hat{\rho} = \left(\frac{\|y^{m+1} - y^m\|}{\|y^1 - y^0\|} \right)^{1/m} \quad (.7)$$

where the norms used in above error estimates are scaled norms which depend on the error tolerances specified by the user. The iteration of the Newton method has converged when

$$\frac{\hat{\rho}}{1 - \hat{\rho}} \|y^{m+1} - y^m\| < 0.3 \quad (.8)$$

If $\hat{\rho} > 0.9$ or $m > 4$ and the iteration is still not converged, then the step size is lowered and/or the iteration matrix based on the current calculations of y, y' and $\hat{\alpha}_2$

is computed and then step size again attempted. To chose the order of the back-ward Euler formula the algorithm of DASSL estimates what the error would have been if the last few steps had been taken at constant step size, at the current oder k , and at $k - 2$, $k - 1$ and $k + 1$. If the estimates at these steps increase as k increases then the order k of the backward Euler formula is decreased otherwise the order is increased. The new time step Δt_{n+1} is chosen such that the error estimate based on taking constant step size Δt_n at order k_{n+1} satisfies the error test.

As Newton method converges more rapidly if the initial guess y_n^0 is accurate. DASSL estimates an initial guess for y_n by solving the polynomial which is interpolated by the computed solution at the last $k + 1$, times *i.e.*, t_{n-1} , t_{n-2} , \dots , $t_{n-(k+1)}$ at the current time t_n . And the initial guess for y'_n is calculated by computing the derivative of this polynomial at t_n . After getting the initial guess y_n^0 , Newton method is used to solve for y_n as in (.5) except that now the derivative is computed by the k th order back-ward Euler formula.

To solve the linear systems of equations $A\mathbf{x} = b$ arising at each successive time step of Newton iteration, DASSL uses a subroutine package LINPACK [28]. LINPACK is a package that uses direct methods such as Gaussian elimination, Cholesky decomposition, QR and singular value decomposition methods to solve the linear systems of equations.

Appendix B: The expressions for artificial right-hands-sides of Example 1,2 and 3

The expressions for F_u , F_v , F_ψ and F_c on the right-hand-side of example 1 are given in this appendix.

$$\begin{aligned}
 F_u(\mathbf{x}, t) = & \rho \left(\frac{1}{2\pi^2} e^{(t-1)} \sin^2(x) y \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right) + \frac{1}{2\pi^4} e^{2(t-1)} \sin^3(x) \right. \\
 & \times y^2 \left(1 - \frac{y}{2\pi}\right)^2 \left(1 - \frac{y}{\pi}\right)^2 \cos(x) - \frac{1}{2\pi^2} e^{(t-1)} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 \\
 & \left. \left(\frac{1}{2\pi^2} e^{(t-1)} \sin^2(x) \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right) - \frac{1}{4\pi^3} e^{(t-1)} \right. \right. \\
 & \times \sin^2(x) y \left(1 - \frac{y}{\pi}\right) - \frac{1}{2\pi^3} e^{(t-1)} \sin^2(x) y \left(1 - \frac{y}{2\pi}\right) \left. \left. \right) \right) \\
 & - \mu \left(\frac{e^{(t-1)}}{\pi^2} \cos^2(x) y \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right) - \frac{e^{(t-1)}}{\pi^2} \sin^2(x) y \left(1 - \frac{y}{2\pi}\right) \right. \\
 & \times \left(1 - \frac{y}{\pi}\right) - \frac{e^{(t-1)}}{2\pi^3} \sin^2(x) \left(1 - \frac{y}{\pi}\right) - \frac{e^{(t-1)}}{\pi^3} \sin^2(x) \left(1 - \frac{y}{2\pi}\right) \\
 & + \frac{1}{2\pi^4} e^{(t-1)} \sin^2(x) y \left. \right) - K_1 \left(-\frac{e^{(t-1)}}{2\pi^2} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 \right. \\
 & - \frac{1}{2\pi^2} e^{(t-1)} \sin^2(x) y \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right) \\
 & \left. - \frac{1}{2} K_2 e^{(t-1)} (\cos(x) \cos(y) + 1), \right.
 \end{aligned}$$

$$\begin{aligned}
F_v(\mathbf{x}, t) = & \left(-\frac{1}{2\pi^2} e^{(t-1)} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 + \frac{1}{2\pi^2} e^{(t-1)} \sin^2(x) y \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right) \right. \\
& \left(-\frac{1}{2\pi^2} e^{(t-1)} \cos^2(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 + \frac{1}{2\pi^2} e^{(t-1)} \sin^2(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 \right) \\
& - \frac{1}{2\pi^2} e^{(t-1)} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 \left(-e^{(t-1)} / \pi^2 \sin(x) \cos(x) y \left(1 - \frac{y}{2\pi}\right)^2 \right. \\
& \left. \left. + \frac{1}{2\pi^3} e^{(t-1)} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right) \right) \right) - e^{(t-1)} \sin(y) - \\
& \mu \left(2e^{(t-1)} / \pi^2 \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 - \frac{e^{(t-1)}}{\pi^2} \sin(x) \cos(x) \left(1 - \frac{y}{2\pi}\right)^2 \right. \\
& \left. + \frac{2e^{(t-1)}}{\pi^3} \sin(x) \cos(x) y \left(1 - \frac{y}{2\pi}\right) - \frac{1}{4\pi^4} e^{(t-1)} \sin(x) \cos(x) y^2 \right) \\
& - \frac{2K_3}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1) - K_4 \left(\frac{1}{2\pi^2} e^{(t-1)} \sin^2(x) y \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right) \right. \\
& \left. + \frac{1}{2\pi^2} e^{(t-1)} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 \right) - \frac{K_5}{2} e^{(t-1)} (\cos(x) \cos(y) + 1),
\end{aligned}$$

$$\begin{aligned}
F_\psi(\mathbf{x}, t) = & \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) + \epsilon e^{(t-1)} \cos(x) \cos(y) \\
& + \gamma \left(\left(1 - \frac{2}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1) \right) (W_a(e^{(t-1)} (\cos(x) \cos(y) + 1)) \right. \\
& + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 - \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \\
& + \frac{15}{2} L_a T_a e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1)\right)^2 \\
& + \frac{2}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1) (W_b(e^{(t-1)} (\cos(x) \cos(y) + 1)) \\
& + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 - \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \\
& \left. \left. + \frac{15}{2} L_b T_b e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1)\right)^2 \right) \right),
\end{aligned}$$

$$F_c(\mathbf{x}, t) = F_{1c}(\mathbf{x}, t) + F_{2c}(\mathbf{x}, t).$$

where

$$\begin{aligned}
F_{1c}(\mathbf{x}, t) = & \frac{2}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1) + \frac{1}{2\pi^2} e^{(t-1)} \sin^2(x) y \left(1 - \frac{y}{2\pi}\right) \left(1 - \frac{y}{\pi}\right) \left(\frac{4}{\pi^2} e^{(t-1)} x \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1) - \frac{2}{\pi^3} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right) (\cos(y) + 1)\right) \\
& + \frac{e^{2(t-1)}}{\pi^4} \sin(x) \cos(x) y^2 \left(1 - \frac{y}{2\pi}\right)^2 x^2 \left(1 - \frac{x}{2\pi}\right)^2 \sin(y) \\
& - \left(\frac{15}{2} (D_L - D_S) e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1)\right)^2 \right. \\
& + \alpha \left(D_S + \frac{1}{8} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \left(10 - \frac{15}{2} e^{(t-1)} (\cos(x) \cos(y) + 1)\right) \right. \\
& + \left.\frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 (D_L - D_S)\right) \\
& \left. \left(1 - \frac{4}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1)\right) \left(W_b \left(e^{(t-1)} (\cos(x) \cos(y) + 1) + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \right. \right. \right. \\
& - \left.\frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2\right) + \frac{15}{2} L_b T_b e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1)\right)^2 \\
& - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \left. \right)^2 - W_a \left(e^{(t-1)} (\cos(x) \cos(y) + 1) + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \right. \\
& - \left.\frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2\right) - \frac{15}{2} L_a T_a e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 - \frac{1}{2} \exp^{(t-1)} (\cos(x) \cos(y) + 1)\right)^2 \left. \right) \left(\frac{-1}{2} e^{(t-1)} \sin(x) \cos(y) \left(\frac{4}{\pi^2} e^{(t-1)} x \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1) \right. \right. \\
& - \left.\frac{2}{\pi^3} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right) (\cos(y) + 1)\right) + \frac{e^{2(t-1)}}{\pi^2} \cos(x) \sin^2(y) x^2 \left(1 - \frac{x}{2\pi}\right)^2 \left. \right) - \left(D_S + \frac{1}{8} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \left(10 - \frac{15}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \right. \right. \\
& + \left.\frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 (D_L - D_S)\right) \left(\frac{4}{\pi^2} e^{(t-1)} \left(1 - \frac{x}{2\pi}\right)^2 (\cos(y) + 1) - \frac{8}{\pi^3} e^{(t-1)} x \left(1 - \frac{x}{2\pi}\right) (\cos(y) + 1) + \frac{1}{\pi^4} e^{(t-1)} x^2 (\cos(y) + 1) \right. \\
& - \left.\left.\frac{2}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi}\right)^2 \cos(y)\right). \right.
\end{aligned}$$

$$\begin{aligned}
F_{2c}(\mathbf{x}, t) = & -\alpha \left(\frac{15}{2} (D_L - D_S) \left(\frac{2}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi} \right)^2 (\cos(y) + 1) \right. \right. \\
& - \frac{4}{\pi^4} e^{2(t-1)} x^4 \left(1 - \frac{x}{2\pi} \right)^4 (\cos(y) + 1)^2 \Big) e^{2(t-1)} (\cos(x) \cos(y) \\
& + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \right)^2 \left(W_b \left(e^{(t-1)} (\cos(x) \cos(y) + 1) \right) \right. \\
& + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 - \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \Big) \\
& + \frac{15}{2} L_b T_b e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \right)^2 \\
& - W_a \left(e^{(t-1)} (\cos(x) \cos(y) + 1) + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \right. \\
& - \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \Big) - \frac{15}{2} L_a T_a e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 \right. \\
& - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \Big)^2 \Big) + \left(D_S + \frac{1}{8} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \left(10 - \right. \right. \\
& \frac{15}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) + \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) \\
& + 1)^2 \Big) (D_L - D_S) \Big) \left(\frac{2}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi} \right)^2 (\cos(y) + 1) \right. \\
& - \frac{4}{\pi^4} e^{2(t-1)} x^4 \left(1 - \frac{x}{2\pi} \right)^4 (\cos(y) + 1)^2 \Big) \left(2(W_b - W_a) \left(1 + \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) \right. \right. \\
& + 1)^2 - 3e^{(t-1)} (\cos(x) \cos(y) + 1) \Big) + 60(L_b T_b - L_a T_a) \left(\frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) \right. \\
& + 1) + \frac{1}{4} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 - \frac{3}{4} e^{2(t-1)} (\cos(x) \cos(y) \\
& + 1)^2 \Big) \Big) \left(\frac{1}{4} e^{2(t-1)} \sin^2(x) \cos^2(y) + \frac{1}{4} e^{2(t-1)} \cos^2(x) \sin^2(y) \right) \\
& + \alpha * \left(D_S + \frac{1}{8} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \left(10 - \frac{15}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \right. \right. \\
& + \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \Big) (D_L - D_S) \Big) \left(\frac{2}{\pi^2} e^{(t-1)} x^2 \left(1 - \frac{x}{2\pi} \right)^2 (\cos(y) + 1) \right. \\
& - \frac{4}{\pi^4} e^{2(t-1)} x^4 \left(1 - \frac{x}{2\pi} \right)^4 (\cos(y) + 1)^2 \Big) \left(W_b \left(e^{(t-1)} (\cos(x) \cos(y) + 1) \right) \right. \\
& + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 - \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \Big) \\
& + \frac{15}{2} L_b T_b e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \right)^2 \\
& - W_a \left(e^{(t-1)} (\cos(x) \cos(y) + 1) + \frac{1}{2} e^{3(t-1)} (\cos(x) \cos(y) + 1)^3 \right. \\
& - \frac{3}{2} e^{2(t-1)} (\cos(x) \cos(y) + 1)^2 \Big) - \frac{15}{2} L_a T_a e^{2(t-1)} (\cos(x) \cos(y) \\
& + 1)^2 \left(1 - \frac{1}{2} e^{(t-1)} (\cos(x) \cos(y) + 1) \right)^2 \Big) e^{(t-1)} \cos(x) \cos(y).
\end{aligned}$$

Right-hand-side expressions for the Example 2 are given below:

$$\begin{aligned}
F_u(\mathbf{x}, t) = & \rho(4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y) \\
& + 4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y)(8\pi e^{(t-1)}x(1-x)^2\sin(2\pi y)\cos(2\pi y) \\
& - 8\pi e^{(t-1)}x^2(1-x)\sin(2\pi y)\cos(2\pi y)) - 2e^{(t-1)}x(2x^2 - 3x \\
& + 1)\sin(2\pi y)^2(8\pi^2 e^{(t-1)}x^2(1-x)^2\cos(2\pi y)^2 - 8\pi^2 e^{(t-1)}x^2(1 \\
& - x)^2\sin(2\pi y)^2)) - 2e^{(t-1)}\sin(2\pi x)\pi - \mu(8\pi e^{(t-1)}(1-x)^2\sin(2\pi y)\cos(2\pi y) \\
& - 32\pi e^{(t-1)}x(1-x)\sin(2\pi y)\cos(2\pi y) + 8\pi e^{(t-1)}x^2\sin(2\pi y)\cos(2\pi y) \\
& - 64\pi^3 e^{(t-1)}x^2(1-x)^2\cos(2\pi y)\sin(2\pi y)) - \frac{K_1}{4}e^{(t-1)}(\cos(2\pi x) + \\
& \cos(2\pi y) + 2)(-2e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2 \\
& - 4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y)) \\
& - \frac{K_2}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2),
\end{aligned}$$

$$\begin{aligned}
F_v(\mathbf{x}, t) = & \rho(-2e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2 + 4\pi e^{(t-1)}x^2(1 \\
& - x)^2\sin(2\pi y)\cos(2\pi y)(-2e^{(t-1)}(2x^2 - 3x + 1)\sin(2\pi y)^2 \\
& - 2e^{(t-1)}x(4x - 3)\sin(2\pi y)^2) + 16e^{2(t-1)}x^2(2x^2 - 3x \\
& + 1)^2\sin(2\pi y)^3\cos(2\pi y)\pi) - \mu(-4e^{(t-1)}(4x - 3)\sin(2\pi y)^2 \\
& - 8e^{(t-1)}x\sin(2\pi y)^2 - 16e^{(t-1)}x(2x^2 - 3x + 1)\cos(2\pi y)^2\pi^2 \\
& + 16e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2\pi^2) - 2K_3e^{2(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)(x^2(1-x)^2 + y^2(1-y)^2) - \frac{K_4}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)(4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y) + 2e^{(t-1)}x(2x^2 \\
& - 3x + 1)\sin(2\pi y)^2) - \frac{K_5}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2),
\end{aligned}$$

$$\begin{aligned}
F_\psi(\mathbf{x}, t) = & \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) - \epsilon(-e^{(t-1)}\cos(2\pi x)\pi^2 \\
& - e^{(t-1)}\cos(2\pi y)\pi^2) + \gamma((1 - 8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2)) \\
& (W_a(\frac{1}{2}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2) \\
& + \frac{15}{8}L_aT_a e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2))^2) + 8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2)(W_b(\frac{1}{2}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2) + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^2) + \frac{15}{8}L_bT_b e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 \\
& - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2))^2),
\end{aligned}$$

$$F_c(\mathbf{x}, t) = F_{1c}(\mathbf{x}, t) + F_{2c}(\mathbf{x}, t).$$

where

$$\begin{aligned}
F_{1c}(\mathbf{x}, t) = & 8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2) + 32\pi e^{2(t-1)}x^2(1 \\
& -x)^2 \sin(2\pi y) \cos(2\pi y)(2x(1-x)^2 - 2x^2(1-x)) - 16e^{2(t-1)}x(2x^2 \\
& - 3x + 1) \sin(2\pi y)^2(2y(1-y)^2 - 2y^2(1-y)) - \left(\frac{15}{8}(D_L \right. \\
& - D_S)e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2))^2 + \alpha(D_S + \frac{1}{64}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3(10 \\
& - \frac{15}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2)(D_L - D_S))(1 - 16e^{(t-1)}(x^2(1-x)^2 + y^2(1 \\
& - y)^2))(W_b(\frac{1}{2}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2) \\
& + \frac{15}{8}L_bT_b e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2))^2 - W_a(\frac{1}{2}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) \\
& + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2) - \frac{15}{8}L_aT_a e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2))^2)(-4e^{2(t-1)}\sin(2\pi x)\pi(2x(1-x)^2 - 2x^2(1-x)) \\
& - 4e^{2(t-1)}\sin(2\pi y)\pi(2y(1-y)^2 - 2y^2(1-y))) - (D_S \\
& + \frac{1}{64}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3(10 - \frac{15}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2) + \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2)(D_L \\
& - D_S))(8e^{(t-1)}(2(1-x)^2 - 8x(1-x) + 2x^2) + 8e^{(t-1)}(2(1-y)^2 \\
& - 8y(1-y) + 2y^2)) - \alpha(\frac{15}{8}(D_L - D_S)(8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2) \\
& - 64e^{2(t-1)}(x^2(1-x)^2 + y^2(1-y)^2)^2)e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 \\
& - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2))^2(W_b(\frac{1}{2}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2) + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^2) + \frac{15}{8}L_bT_b e^{(t-1)}(t-1)^2(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2))^2.
\end{aligned}$$

$$\begin{aligned}
F_{2c}(\mathbf{x}, t) = & W_a \left(\frac{1}{2} e^{(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{1}{16} e^{3(t-1)} (\cos(2\pi x) \right. \\
& + \cos(2\pi y) + 2)^3 - \frac{3}{8} e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^2 \Big) \\
& - \frac{15}{8} L_a T_a e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^2 \left(1 - \frac{1}{4} e^{(t-1)} (\cos(2\pi x) \right. \\
& + \cos(2\pi y) + 2) \Big)^2 + \left(D_S + \frac{1}{64} e^{3(t-1)} (\cos(2\pi x) + \cos(2\pi y) \right. \\
& + 2)^3 \left(10 - \frac{15}{4} e^{(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{3}{8} e^{2(t-1)} (\cos(2\pi x) \right. \\
& + \cos(2\pi y) + 2)^2 \Big) (D_L - D_S) \Big) (8e^{(t-1)} (x^2(1-x)^2 + y^2(1-y)^2) \\
& - 64e^{2(t-1)} (x^2(1-x)^2 + y^2(1-y)^2)^2) (2(W_b - W_a) (1 + \\
& \frac{3}{8} e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^2 - \frac{3}{2} e^{(t-1)} (\cos(2\pi x) \\
& + \cos(2\pi y) + 2) + 60(L_b T_b - L_a T_a) \left(\frac{1}{4} e^{(t-1)} (\cos(2\pi x) \right. \\
& + \cos(2\pi y) + 2) + \frac{1}{32} e^{3(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^3 \\
& - \frac{3}{16} e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^2 \Big) \left(\frac{1}{4} e^{2(t-1)} \sin(2\pi x)^2 \pi^2 \right. \\
& + \frac{1}{4} e^{2(t-1)} \sin(2\pi y)^2 \pi^2) - \alpha \left(D_S + \frac{1}{64} e^{3(t-1)} (\cos(2\pi x) \right. \\
& + \cos(2\pi y) + 2)^3 \left(10 - \frac{15}{4} e^{(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2) \right. \\
& + \frac{3}{8} e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^2 \Big) (D_L - D_S) \Big) (8e^{(t-1)} (x^2(1-x)^2 \\
& + y^2(1-y)^2) - 64e^{2(t-1)} (x^2(1-x)^2 + y^2(1-y)^2)^2) \\
& \times \left(W_b \left(\frac{1}{2} e^{(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{1}{16} e^{3(t-1)} (\cos(2\pi x) \right. \right. \\
& + \cos(2\pi y) + 2)^3 - \frac{3}{8} e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^2 \Big) \\
& + \frac{15}{8} L_b T_b e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^2 \left(1 - \frac{1}{4} e^{(t-1)} (\cos(2\pi x) \right. \\
& + \cos(2\pi y) + 2) \Big)^2 - W_a \left(\frac{1}{2} e^{(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2) \right. \\
& + \frac{1}{16} e^{3(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8} e^{2(t-1)} (\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^2 \Big) - \frac{15}{8} L_a T_a e^{2(t-1)} (\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2 \left(1 - \frac{1}{4} e^{(t-1)} (\cos(2\pi x) + \cos(2\pi y) + 2) \Big)^2 \Big) \\
& \left(-e^{(t-1)} \cos(2\pi x) \pi^2 - e^{(t-1)} \cos(2\pi y) \pi^2 \right).
\end{aligned}$$

Now we shall give the expressions for the right-hand-side of the anisotropic problem below:

$$\begin{aligned}
F_u(\mathbf{x}, t) = & \rho(4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y) \\
& + 4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y)(8\pi e^{(t-1)}x(1-x)^2\sin(2\pi y)\cos(2\pi y) \\
& - 8\pi e^{(t-1)}x^2(1-x)\sin(2\pi y)\cos(2\pi y)) \\
& - 2e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2(8\pi^2 e^{(t-1)}x^2(1-x)^2\cos(2\pi y)^2 \\
& - 8\pi^2 e^{(t-1)}x^2(1-x)^2\sin(2\pi y)^2)) \\
& - 2e^{(t-1)}\sin(2\pi x)\pi - \mu(8\pi e^{(t-1)}(1-x)^2\sin(2\pi y)\cos(2\pi y) \\
& - 32\pi e^{(t-1)}x(1-x)\sin(2\pi y)\cos(2\pi y) \\
& + 8\pi e^{(t-1)}x^2\sin(2\pi y)\cos(2\pi y) - 64\pi^3 e^{(t-1)}x^2(1-x)^2\cos(2\pi y)\sin(2\pi y)) \\
& - \frac{K_1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)(-2e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2 \\
& - 4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y)) \\
& - \frac{K_2}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)
\end{aligned}$$

$$\begin{aligned}
F_v(\mathbf{x}, t) = & \rho(-2e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2 \\
& + 4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y) - 2e^{(t-1)}(2x^2 - 3x \\
& + 1)\sin^2(2\pi y) - 2e^{(t-1)}x(4x - 3)\sin^2(2\pi y)) + 16e^{2(t-1)}x^2(2x^2 \\
& - 3x + 1)^2\sin(2\pi y)^3\cos(2\pi y)\pi) \\
& - \mu(-4e^{(t-1)}(4x - 3)\sin(2\pi y)^2 - 8e^{(t-1)}x\sin(2\pi y)^2 \\
& - 16e^{(t-1)}x(2x^2 - 3x + 1)\cos(2\pi y)^2\pi^2 \\
& + 16e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2\pi^2) - 2K_3e^{2(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)(x^2(1-x)^2 + y^2(1-y)^2) \\
& - \frac{K_4}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)(4\pi e^{(t-1)}x^2(1-x)^2\sin(2\pi y)\cos(2\pi y) \\
& + 2e^{(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^2) \\
& - \frac{K_5}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)
\end{aligned}$$

$$\begin{aligned}
F_\psi(\mathbf{x}, t) = & \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) - 2\pi^2 e^{2(t-1)}x^2(1-x)^2 \\
& \sin(2\pi y)\cos(2\pi y)\sin(2\pi x) + e^{2(t-1)}x(2x^2 - 3x + 1)\sin(2\pi y)^3\pi \\
& - \gamma ep0(1 + \gamma_0\cos(kk * \arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)})))(-e^{(t-1)}\cos(2\pi x)\pi^2 \\
& - e^{(t-1)}\cos(2\pi y)\pi^2) + \gamma((1 - 8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2)) \\
& \times (W_a(1/2e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2) \\
& + \frac{15}{8}L_aT_a e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 \\
& - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2))^2) + 8e^{(t-1)}(x^2(1-x)^2 \\
& + y^2(1-y)^2)(W_b(1/2e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) \\
& + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^2) + \frac{15}{8}L_bT_b e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2))^2)) \\
& + \gamma(\frac{-1}{2}ep0^2(1 + \gamma_0\cos(kk * \arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)})))\gamma_0kk^2 \\
& \cos(kk\arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)})) + \frac{1}{2}ep0^2\gamma_0^2kk^2\sin(kk\arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)}))^2) \\
& \times (e^{(t-1)}\cos(2\pi x)\pi^2 + e^{(t-1)}\cos(2\pi y)\pi^2 - (-e^{(t-1)}\cos(2\pi y)\pi^2 \\
& + e^{(t-1)}\cos(2\pi x)\pi^2)\cos(2 * \arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)}))) \\
& + \gamma ep0^2(1 + \gamma_0\cos(kk * \arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)}))) \\
& \times \gamma_0kk\sin(kk\arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)}))\sin(2\arctan(\frac{\sin(2\pi y)}{\sin(2\pi x)})) \\
& \times (-e^{(t-1)}\cos(2\pi y)\pi^2 + e^{(t-1)}\cos(2\pi x)\pi^2)
\end{aligned}$$

$$F_c(\mathbf{x}, t) = F_{1c}(\mathbf{x}, t) + F_{2c}(\mathbf{x}, t).$$

where

$$\begin{aligned}
F_{1c}(\mathbf{x}, t) = & 8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2) + 32\pi e^{2(t-1)}x^2(1-x)^2 \sin(2\pi y) \\
& \times \cos(2\pi y)(2x(1-x)^2 - 2x^2(1-x)) - 16e^{2(t-1)}x(2x^2 - 3x \\
& + 1)\sin(2\pi y)^2(2y(1-y)^2 - 2y^2(1-y)) - \left(\frac{15}{8}(D_L - D_S)e^{2(t-1)}\right. \\
& \times (\cos(2\pi x) + \cos(2\pi y) + 2)^2 \left(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) \right. \\
& \left. + 2)\right)^2 + \alpha \left(D_S + \frac{1}{64}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 \left(10 \right. \right. \\
& \left. \left. - \frac{15}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) \right. \right. \\
& \left. \left. + \cos(2\pi y) + 2)^2\right)(D_L - D_S)\right) \times \left(1 - 16e^{(t-1)}(x^2(1-x)^2 + y^2(1 \right. \\
& \left. - y)^2)\right) \left(W_b\left(\frac{1}{2}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{1}{16}e^{3(t-1)}\right. \right. \\
& \times (\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& \left. + 2)^2\right) + \frac{15}{8}L_bT_b e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2 \left(1 - \frac{1}{4}e^{(t-1)}\right. \\
& \times (\cos(2\pi x) + \cos(2\pi y) + 2))^2 - W_a \left(1/2e^{(t-1)}(\cos(2\pi x) \right. \\
& \left. + \cos(2\pi y) + 2) + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 \right. \\
& \left. - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2\right) - \frac{15}{8}L_aT_a e^{2(t-1)}(\cos(2\pi x) \\
& \left. + \cos(2\pi y) + 2)^2 \left(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)\right)^2\right) \\
& \left(-4e^{2(t-1)}\sin(2\pi x)\pi(2x(1-x)^2 - 2x^2(1-x)) - 4e^{2(t-1)}\right. \\
& \times \sin(2\pi y)\pi(2y(1-y)^2 - 2y^2(1-y))) - \left(D_S + \frac{1}{64}e^{3(t-1)}\right. \\
& \times (\cos(2\pi x) + \cos(2\pi y) + 2)^3 \left(10 - \frac{15e^{(t-1)}}{4}(\cos(2\pi x) + \cos(2\pi y) \right. \\
& \left. + 2) + \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2\right)(D_L - D_S)\right) \left(8e^{(t-1)}\right. \\
& \times (2(1-x)^2 - 8x(1-x) + 2x^2) + 8e^{(t-1)}(2(1-y)^2 - 8y(1-y) \\
& \left. + 2y^2)\right) - \alpha \left(\frac{15}{8}(D_L - D_S)(8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2) \right. \\
& \left. - 64e^{2(t-1)}(x^2(1-x)^2 + y^2(1-y)^2)^2\right) e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& \left. + 2)^2 \left(1 - \frac{e^{(t-1)}}{4}(\cos(2\pi x) + \cos(2\pi y) + 2)\right)^2 \left(W_b\left(\frac{e^{(t-1)}}{2}(\cos(2\pi x) \right. \right. \right. \\
& \left. \left. + \cos(2\pi y) + 2)\right).
\end{aligned}$$

$$\begin{aligned}
F_{2c}(\mathbf{x}, t) = & \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 \\
& - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2 + \frac{15}{8}L_bT_b e^{2(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2))^2 \\
& - W_a(1/2e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2) - \frac{15}{8}L_aT_a e^{2(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2))^2 \\
& + (D_S + \frac{1}{64}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3(10 - 15/4e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2) + \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2)(D_L \\
& - D_S))(8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2) - 64e^{2(t-1)}(x^2(1-x)^2 \\
& + y^2(1-y)^2)^2)(2(W_b - W_a)(1 + \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2 \\
& - 3/2e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)) + 60(L_bT_b \\
& - L_aT_a)(\frac{1}{4}e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) + 1/32e^{3(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2)^3 - 3/16e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2))(\frac{1}{4}e^{2(t-1)}\sin(2\pi x)^2\pi^2 + \frac{1}{4}e^{2(t-1)}\sin(2\pi y)^2\pi^2) - \alpha(D_S \\
& + \frac{1}{64}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3(10 - 15/4e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2) + \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2)(D_L \\
& - D_S))(8e^{(t-1)}(x^2(1-x)^2 + y^2(1-y)^2) - 64e^{2(t-1)}(x^2(1-x)^2 \\
& + y^2(1-y)^2)^2)(W_b(1/2e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) \\
& + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2) + \frac{15}{8}L_bT_b e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2))^2 - W_a(1/2e^{(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2) \\
& + \frac{1}{16}e^{3(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^3 - \frac{3}{8}e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) \\
& + 2)^2) - \frac{15}{8}L_aT_a e^{2(t-1)}(\cos(2\pi x) + \cos(2\pi y) + 2)^2(1 - \frac{1}{4}e^{(t-1)}(\cos(2\pi x) \\
& + \cos(2\pi y) + 2))^2)(-e^{(t-1)}\cos(2\pi x)\pi^2 - e^{(t-1)}\cos(2\pi y)\pi^2).
\end{aligned}$$

Bibliography

- [1] A. A. Wheeler, W. J. Boettinger and G. B. McFadden, “Phase-field model for isothermal phase transitions in binary alloys”, *Physical Review A*, Vol. **45**, No. 10, pp. 7424-7439, 1992.
- [2] A. A. Wheeler, W. J. Boettinger and G. B. McFadden, “Phase-field model of solute trapping during solidification”, *Physical Review E*, Vol. **47**, No. 3, pp. 1893-1909, 1993.
- [3] A. Belmiloudi, “Method of characteristics and error estimates of the perturbation of given mean flow. Application of mathematics in Engineering and Business Sozopol”, *Proceedings of the XXII Summer School*, pp. 25-38, 1996.
- [4] A. Belmiloudi and F. Brossier, “Regularity results for a Navier-Stokes type problems related to oceanography”, *Acta Applicandae Mathematicae*, Vol **48**, pp. 299-316, 1997.
- [5] A. Belmiloudi, “Robin-type boundary control problems for the non-linear Boussinesq type equations”, *J. Math. Anal. Appl.*, **273**, pp. 428-456, 2002.
- [6] A. Belmiloudi, “Robust and optimal control problems to a phase-field model for the solidification of a binary alloy with a constant temperature”, *J. dynamical and control systems*, Vol. **10**, No. 4, pp. 453-499, 2004.
- [7] A. Belmiloudi and J. P. Yvon, “Robust control of a non-isothermal solidification model”, *WSEAS Transactions on systems*, Vol. **4**, No. 12, pp. 2291-2300, 2005.
- [8] A. Belmiloudi, “Stabilization, optimal and robust control”, *Springer*, 2008.
- [9] A. Visintin, “Models of Phase Transitions”, *Birkhauser Boston*, 1996.
- [10] B. Kaouil, M. Nouredine, R. Nassif, Y. Boughaleb, ”Phase-field modelling of dendritic growth behaviour towards the cooling/heating of pure nickel”, *M. J. Condensed Matter*, Vol. **6**, No. 1, pp. 109-112, 2005.

- [11] C. Beckermann, H. J. Diepers, I. Steinbach, A. Karma and X. Tong, "Modeling melt convection in phase-field simulations of solidification", *J. Comput. Phys.*, Vol **154**, pp. 468-496, 1999.
- [12] C. W. Gear, "Simulations numerical solution of differential/algebraic equations", *IEEE Trans. on Circuit Theory*, CT-18, No. 1, pp. 89-95, 1971.
- [13] D. Jacqmin, "Calculation of two-phase Navier-Stokes flows using phase-field modeling", *J. Comput. Phys.*, Vol. **155**, pp. 96-127, 1999.
- [14] D. Kessler, "Modeling, Mathematical and numerical study of a solutal phase-field model", Thèse, Lausanne EPFL, 2001.
- [15] D. M. Anderson, G. B. McFadden and A. A. Wheeler, "A phase-field model of solidification with convection", *Physica D*, Vol **135**, pp. 175-194, 2000.
- [16] E. Bansch and A. Schmidt, "Simulation of dendritic crystal growth with thermal convection", *EMS, Interfaces and Free Boundaries*, Vol. **2**, No. 1, pp. 95-115, 2000.
- [17] E. Süli, "Convergence and non-linear stability of Lagrange-Galerkin method for the Navier-Stokes equations.", *Numer. Math.*, Vol. **53**, pp. 459-483, 1988.
- [18] F. Abergel and R. Temam, "On Some Control Problems in Fluid Dynamics", *Theoret. Comput. Fluid Dynamics*, Vol. **1**, pp. 303-325, 1990.
- [19] G. Caginalp, "Stefan and Hele-Shaw type models as asymptotic limits of the phase-field equations", *Physical Review A*, Vol. **39**, No. 11, pp. 5887-5896, 1989.
- [20] G. Tryggvason, A. Esmaeeli and N. Al-Rawahi "Direct numerical simulations of flows with phase change", *Computers and Structures*, Vol **83**, pp. 445-453, 2005.
- [21] H. B. Hadid, D. Henry and S. Kaddeche, "Numerical study of convection in the horizontal Bridgman configuration under the action of a constant magnetic field. Part 1. Two dimensional flow", *J. Fluid Mech.*, Vol. **333**, pp. 23-56, 1997.
- [22] H. B. Hadid and D. Henry, "Numerical study of convection in the horizontal Bridgman configuration under the action of a constant magnetic field. Part 2. Three dimensional flow", *J. Fluid Mech.* Vol. **333**, pp. 57-83, 1997.
- [23] H. Brezis, "Analyse fonctionnelle", Masson, Paris, 1983.

- [24] J. A. Warren and W. J. Boettinger, "Prediction of dendritic growth and microsegregation patterns in a binary alloy using the phase-field method", *Acta metall. mater*, Vol. **43**, No. 2, pp. 689-703, 1995.
- [25] J. C. Ramizer, C. Beckermann, A. Karma and H. J. Diepers, "Phase-field modeling of binary alloy solidification with couple heat and solute diffusion", *Physical Review E*, Vol **69**, pp. (051607-1)-(051607-16), 2004.
- [26] J. C. Ramizer and C. Beckermann, "Examination of binary alloy free dendritic growth theories with a phase-field model", *Acta Materialia*, Vol **53**, pp. 1721-1736, 2005.
- [27] J. E. Taylor and J. W. Cahn, "Diffusive interfaces with sharp corners and facets: Phase field models with strongly anisotropic surfaces", *Physica D*, Vol. **112**, pp. 381-411, 1998.
- [28] J. J. Dongarra, J. R. Bunch, C. B. Moler and G. W. Stewart, "LINPACK Users Guide", *SIAM, Philadelphia*, 1979.
- [29] J. K. Roplekar and J.A. Dantzig, "A study of solidification with a rotating magnetic field", *Int. J. Cast Met. Res.* Vol. **14**, pp. 79-95, 2001.
- [30] J. L. Lions, E. Magenes, "Nonhomogeneous boundary value problems and applications", Springer-Verlag, New York, 1972.
- [31] J. L. Lions, "Quelques méthodes de resolution des problèmes aux limites non-linéaire", Gauthier-Villars, Paris, 1969.
- [32] J. L. Lions, "Contrôle optimale de systèmes gouvernés par des équations aux dérivées partielles", Dunod, Paris, 1968.
- [33] J. Narski and M. Picasso, "Adaptive 3D finite elements with high aspect ratio for dendritic growth of a binary alloy including fluid flow induced by shrinkage", *Fluid Dynamics and Materials Processing*, Vol. **1**, No. 1, pp. 1-13, 2006.
- [34] J. Rappaz and J. F. Scheid, "Existence of solutions to a phase-field model for the isothermal solidification process of a binary alloy", *Mathematical Methods in the Applied Sciences*, Vol. **23**, pp. 491-513, 2000.
- [35] J. Rappaz and R. Touzani, "Numerical simulation of solidification processes in electromagnetic casting", *Magnetohydrodynamics in Process Metallurgy*, Eds J. Szekely, J. W. Evans, K. Blasek, N. El-Kaddah. TMS, pp. 181-185, 1992.

- [36] K. H. Hoffman and L. Jiang, "Optimal control of a phase-field model for solidification", *Numer. Funct. Anal. Optim.*, Vol. **13**, pp. 1127, 1992.
- [37] L. R. Petzold, "A discription of DASSL: A differential/algebraic system solver", *Scientific computing, IMACS Trans. Sci. Comput.*, pp. 65-68, 1983.
- [38] M. Gunzberger, E. Ozugurlu, J. Turner, H. Zhang, "Controlling transport phenomena in the Czochralski crystal growth process", *J. Crystal Growth*, Vol. **234**, pp. 47-62, 2002.
- [39] M. Grugicic, G. Cao and R. S. Millar, "Computer modelling of the evolution of dendrite microstructure in binary alloys during non-isotheraml solidification", *J. Materials synthesis and processing*, Vol. **10**, No. 4, pp. 191-203, 2002.
- [40] M. Li, T. Takuya, N. Omura, K. Miwa, "Effects of magnetic field and electric current on the solidification of AZ91D magnesium alloys using an electromagnetic vibration technique", *J. of Alloys and Compounds*, Vol. **487**, pp. 187-193, 2009.
- [41] M. Li, T. Takuya, N. Omura, K. Miwa, "The solidification behavior of the AZ61 magnesium alloy during electromagnetic vibration processing", *J. of Alloys and Compounds*, Vol. **494**, pp. 116-122, 2010.
- [42] M. Watanabe, D. Vizman, J. Friedrich, G. Mueller, "Large modification of crystal-melt interface shape during Si crystal growth by using electromagnetic Czochralski method", *J. Crystal Growth*, Vol. **292**, pp. 252-256, 2006.
- [43] N. A. Ahmad, A. A. Wheeler, W. J. Boettinger and G. B. McFadden, "Solute trapping and solute drag in a phase-field model of rapid solidification", *Physical Review E*, Vol. **58**, No. 3, pp. 3436-3450, 1998.
- [44] N. Al-Rawahi and G. Tryggvason, "Numerical simulation of dendritic solidification with convection: Two dimensional geometry", *J. Comput. Phys.*, Vol **180**, pp. 471-496, 2002.
- [45] N. Al-Rawahi and G. Tryggvason, "Numerical simulation of dendritic solidification with convection: Three dimensional flow", *J. Comput. Phys.*, Vol **194**, pp. 667-696, 2004.
- [46] O. Kruger, "Modélisation et analyse numerique de problèmes de réaction-diffusion provenant de la solidification d'alliages binaires", Thèse No. 2071, Lausanne EPFL, 1999.

- [47] P. G. Ciarlet, “The finite element method for elliptic problems”, North-Holland, Amsterdam, 1989.
- [48] P. J. Prescott and F.P. Incropera, “Magnetically damped convection during solidification of a binary metal alloy”, *Trans. ASME*, Vol. **115**, pp. 302-310, 1993.
- [49] P. Laurencot, “Weak solutions to a phase-field model with non-constant thermal conductivity”, *Quart. Appl. Math.*, Vol. **4**, pp. 739-760, 1997.
- [50] P. N. Brown, A.C. Hindmarsh, and L.R. Petzold, “Using Krylov methods in the solution of large-scale differential-algebraic systems”, *SIAM J. Sci. Comput.*, Vol. **15**, No.6, pp. 1467-1488, 1994.
- [51] P. Zhao, J. C. Heinrich and D. R. Poirier, “Dendritic solidification of binary alloys with free and forced convection”, *Int. J. Numer. Meth. Fluids*, Vol. **49**, pp. 233-266, 2005.
- [52] R. A. Adams, “Sobolev spaces”, Academic, New York, 1975.
- [53] R. Kobayashi, “Modeling and numerical simulations of dendritic crystal growth”, *Physica D*, Vol. **63**, pp. 410-423, 1993.
- [54] R. Sampath, “The adjoint method for the design of directional binary alloy solidification processes in the presence of a strong magnetic field”, Thesis, Cornell University USA, 2001.
- [55] R. Temam, “Navier-Stokes equations”, North-Holland, Amsterdam, 1977.
- [56] R. Tonhardt and G. Amberg, “Simulation of natural convection effects on succinonitrile crystals”, *Physical Review E*, Vol. **62**, No. 1, pp. 828-836, 2000.
- [57] S. L. Wang et al., “Thermodynamically-Consistent Phase-Field Models for Solidification”, *Physica D*, Vol. **69**, pp. 189-200, 1993.
- [58] T. Takaki, T. Fukuoka, Y. Tomita, “Phase-field Simulations during Directional Solidification of a Binary Alloy using Adaptive Finite Element Method”, *J. Crystal Growth*, Vol. **283**, No. 1-2, pp. 263-278, 2005.
- [59] V. Galindo, G. Gerbeth, W. Von Ammon, E. Tomzig and J. Virbulis, “Crystal growth melt flow previous term control next term by means of magnetic fields”, *Energy Conv. Manage.* Vol. **43**, pp. 309-316, 2002.

-
- [60] W. J. Boettinger, J. A. Warren, C. Beckermann and A. Karma, “Phase-field simulation of solidification”, *Annu. Rev. Mater. Res.*, Vol **32**, pp. 163-194, 2002.
 - [61] X. Tong, C. Beckermann, A. Karma and Q. Li, “Phase-field simulations of dendritic crystal growth in a forced flow”, *Physical Review E*, Vol **63**, pp. (061601-1)-(061601-16), 2001.

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